ON THE DETERMINATION OF SAMPLE SIZE

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PREFACE

The problem of obtaining a sample size necessary to obtain an estimate of a parameter with a guaranteed precision has been solved. A two-step technique developed by Graybill enables one to find a sample size such that (1) the probability is $1 - \alpha$, the confidence coefficient, that the confidence interval contains the parameter, and (2) the probability that the width is less than or equal to d specified units is greater than or equal to β^2, the width coefficient. This thesis is a continuation of that technique. Graybill's method is applied in finding necessary sample sizes for precise interval estimates of the ratio of variances from two normal populations and the parameter of the rectangular density. Expected sample sizes for the mean and variance of a normal distribution are computed. The expected sample sizes found with Graybill's technique are compared with those found with other methods.

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TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

In problems of estimating a specified parameter, either a point or an interval estimate is used. In order to find the sample size necessary to obtain these estimates with a guaranteed precision, a two-step procedure is usually required. If a point estimate of a parameter is required with a specified precision, solutions are given by Cox (1), Anscombe (2), and Birnbaum and Healy (3). The former two methods are approximate and rather difficult to apply, whereas the latter seems relatively simple and quite exact. Any one of these three methods will determine the sample size necessary for a point estimate to have the variance of the estimator less than or equal to a specified value.

The interval-estimate approach is considerably different. Stein's (4) procedure is the classic example. A sample of size n is found so that an estimate of the mean of a normal population is made such that a $1 - \alpha$ confidence interval has width less than or equal to a specified length. Graybill's (5) technique provides for a value of n so that the width of a $1 - \alpha$ confidence interval has probability $\geq \beta^2$ of being less than d prescribed units. In this study, this latter technique is enlarged upon. This method is used to obtain the necessary sample size in order to have precise interval estimates of the ratio of variances and the parameter of the rectangular distribution. The formulation for obtaining expected sample sizes for precise width interval estimates of the mean and variance of the normal distribution is developed and comparisons are made with the work of Stein (4) and Birnbaum and Healy (3). Tables of expected sample sizes for desired width interval estimates of various parameters are presented.

The following notation will be used:

$P(A)$	the probability that the event A occurs
w	the width of a confidence interval
θ	the parameter which is to be estimated
$1 - \alpha$	the probability that the confidence interval contains θ
d	the desired width of an interval
β^2	the width coefficient, the probability that $w \leq d$
$f(x) = O[g(x)]$	$[f(x)/g(x)]$ remains bounded as x tends to its limit
$f(x) = o[g(x)]$	$[f(x)/g(x)] \to 0$ as x tends to its limit
$E(x)$	the expected value of x
$\chi^2_\alpha(n)$	a value such that $\int_{\chi^2_\alpha(n)}^{\infty} W(\chi^2;n)\, d\chi^2 = \alpha$, where $W(\chi^2;n)$ is the Chi-square distribution with n - 1 degrees of freedom
$t_\alpha(n)$	a value such that $\int_{t_\alpha(n)}^{\infty} G(t;n)\, dt = \alpha$, where $G(t;n)$ is Student's t distribution with n - 1 degrees of freedom

CHAPTER II

BOUNDS ON THE WIDTH COEFFICIENT

The method given by Graybill (5) describes a method whereby a sample of size n may be obtained so that the following criteria are met: (1) a $1 - \alpha$ confidence interval is placed on the parameter, and (2) the probability that the width of the interval $\leq$ d prescribed units $\geq \beta^2$.

The probability that the width of a confidence interval does not exceed a predetermined d is bounded from above by $(2\beta - \beta^2)$ and from below by β^2.

If the assumptions in Reference (5) hold, we have

$$P(W \leq d) = P\{w \leq h[t(z),n] \mid t(z) \leq \theta\}\ P[t(z) \leq \theta] + P\{w \leq h[t(z),n] \mid t(z) \geq \theta\}\ P[t(z) \geq \theta], \tag{2.1}$$

where

$$d = h[t(z),n], \tag{2.2a}$$

$$P[t(z) \leq \theta] = 1 - \beta, \tag{2.2b}$$

and

$$P[t(z) \geq \theta] = \beta. \tag{2.2c}$$

Suppose θ is real-valued and $t(z) < \theta$. Then as $t(z) \rightarrow -\infty$, we obtain

$$P\{w \leq h[t(z),n] \mid t(z) < \theta,\ t(z) \rightarrow -\infty\} \rightarrow 0. \tag{2.3}$$

This follows from the assumption that $h[t(z),n]$ is monotonically increasing for every n and attains its smallest value as $t(z) \rightarrow -\infty$.

Suppose $t(z) \to \theta$ from either the positive or negative side. It follows that

$$P\{w \leq h[t(z),n] | t(z) \to \theta\} \to P[w \leq h(\theta;n)] = \beta. \tag{2.4}$$

Suppose $t(z) > \theta$, and $t(z) \to \infty$. Since the monotonicity of $h[t(z),n]$ implies that w will be necessarily smaller than the maximum of $h[t(z),n]$ we obtain

$$P\{w \leq h[t(z),n] | t(z) > \theta, t(z) \to \infty\} \to 1. \tag{2.5}$$

Let

$$p_1 = P\{w \leq h[t(z),n] | t(z) < \theta\} \; P[t(z) < \theta],$$

and

$$p_2 = P\{w \leq h[t(z),n] | t(z) > \theta\} \; P[t(z) > \theta].$$

Now the maximum of $p_1 + p_2$ can be found. Since $P[t(z) < \theta] = 1 - \beta$, then p_1 is maximized simultaneously with $P\{w \leq h[t(z),n] | t(z) < \theta\}$. This term is maximized in (2.4) and minimized in (2.3). Thus

$$\text{Max}(p_1) = \beta(1 - \beta),$$

and

$$\text{Min}(p_1) = 0.$$

Similarly, from (2.4) and (2.5) and since $P[t(z) > \theta] = \beta$, we may write

$$\text{Max}(p_2) = 1 \times \beta = \beta,$$

and

$$\text{Min}(p_2) = \beta \times \beta = \beta^2.$$

Hence

$$\text{Max}(p_1 + p_2) = \beta + \beta(1 - \beta) = 2\beta - \beta^2,$$

and

$$\text{Min}(p_1 + p_2) = \beta^2.$$

Thus the original assertion has been shown to be true.

The following table gives some upper and lower bounds for various values of β:

TABLE I

BOUNDS ON β FOR A SPECIFIED WIDTH INTERVAL

β	Upper Bound	Lower Bound
0.80	0.9600	0.6400
0.90	0.9900	0.8100
0.95	0.9975	0.9025
0.99	0.9999	0.9500

Throughout this work, only β^2, the lower bound, will be used as the width coefficient; however, it is important to be aware of the upper bound, $2\beta - \beta^2$.

CHAPTER III

A THEOREM FOR THE EXPECTED VALUE OF SAMPLE SIZES

THEOREM. Let the conditions in Section 2 of Reference (5) hold. In addition, assume the following:

k is the smallest integer such that $n \leq k$ and n satisfies $h[t(z),n] = d$; (3.1a)

$f_1(u)$ is determined if the inequality $h[t(z),u] \leq d$ can be solved explicitly for z, where u takes on values of k, and $f_1(u)$ is monotonic in u for every d; (3.1b)

$g_1(z)$ is the density function of z; (3.1c)

$g_1(z)$ is monotonically decreasing in z for all $z \geq z^*$; (3.1d)

$g_1(z) = O(1/z)^t, \ t \geq 3$; (3.1e)

N is an integer so that for all $n > N$, $g_1[f_1(N)] < 1/[f_1(N)]^t$ for a specified $t \geq 3$; (3.1f)

$\sum_{u=N}^{\infty} 1/[f_1(u)]^{t-1}$ converges; (3.1g)

and

$$c_1 = (t - 1)^{-1}. \tag{3.1h}$$

Then the expected value of k is given by

$$E(k) = 1 + \left[\sum_{u=1}^{\infty}\int_{f_1(u)}^{\infty} g_1(z)\, dz\right] - \lim_{N\to\infty} \frac{c_1 N}{[f_1(n)]^{t-1}}.$$

The necessity of these assumptions will be brought out in the proof;

however, a brief explanation may explain the feasibility of some of the requirements. Suppose the expression $h_1(u) = cz$ where c does not contain z. Then if we solve the inequality as stated in (3.1b) for z, it follows that $f_1(u)$ will be monotonically increasing in u. The monotonicity of $g_1(z)$ in (3.1d) and the magnitude of $g_1(z)$ in (3.1e) allows the existence of N in (3.1f). We shall see later that if (3.1g) holds, then E(k) is finite.

PROOF. Using (3.1b) we may solve the inequality $h[t(z),k] \leq d$ explicitly for z. If we solve $h[t(z),n] = d$ for z, we obtain $z = f_1(n)$.

Let k be the smallest integer $\geq n$, so that $z \leq f_1(k)$.

By definition

$$E(k) = \sum_{u=1}^{\infty} uP(u - 1 < n \leq u). \tag{3.2}$$

In order to simplify (3.2) we derive the following:

$$\begin{aligned}
&\sum_{u=1}^{N} uP[(u - 1) < n \leq u] \\
&\quad = \sum_{u=1}^{N} uP(n \leq u) - \sum_{u=1}^{N} uP[n < (u - 1)] \\
&\quad = \sum_{u=1}^{N-1} uP(n \leq u) + NP(n \leq N) - \sum_{u=1}^{N} uP[n < (u - 1)] \\
&\quad = \sum_{u=1}^{N-1} uP(n \leq u) + NP(n \leq N) - \sum_{y=o}^{N-1} (y + 1)\, P(n < y) \\
&\quad = \sum_{u=1}^{N-1} uP(n \leq u) + NP(n \leq N) - \sum_{y=o}^{N-1} [yP(n < y)] - \sum_{y=o}^{N-1} P(n \leq y) \\
&\quad = N[P(n \leq N)] - \sum_{u=1}^{N-1} [1 - P(n > u)] \\
&\quad = N[P(n \leq N) - 1] + 1 + \sum_{u=1}^{N-1} P(n > u) \\
&\quad = \sum_{u=1}^{N-1} P(n > u) + 1 - NP(n > N).
\end{aligned} \tag{3.3}$$

Hence we may write (3.2) as

$$E(k) = \lim_{N\to\infty} \sum_{u=1}^{N} uP[(u - 1) < n \leq u]$$

$$= \lim_{N\to\infty} [-NP(n > N) + 1 + \sum_{u=1}^{N-1} P(n > u)]. \tag{3.4}$$

Suppose $u = N$. Then $z = f_1(N)$. Let n have the distribution function $p(n)$. Then

$$P(n > N) = \int_N^\infty p(n)\, dn.$$

When $n > N$, the fact that z is monotonically increasing in n, see (3.1b), implies that it is simultaneously true that $z > f_1(N)$. Hence, we may write

$$P(n \geq N) = \int_{f_1(N)}^\infty g_1(z)\, dz. \tag{3.5}$$

Because $g_1(z)$ is monotonically decreasing in the range $z > z^*$ and N has been deliberately selected, see (3.1f), then

$$\int_{f_1(N)}^\infty g_1(z)\, dz \leq \int_{f_1(N)}^\infty \frac{1}{z^t}\, dz$$

for all $n > N$ and $t \geq 3$. Hence

$$\lim_{N\to\infty} N \int_{f_1(N)}^\infty g_1(z)\, dz \leq \lim_{N\to\infty} N \int_{f_1(N)}^\infty \frac{1}{z^t}\, dz$$

$$= \lim_{N\to\infty} \frac{c_1 N}{[f_1(N)]^{t-1}}, \tag{3.6}$$

where $c_1 = (t - 1)^{-1}$.

If (3.6) does not exist, then the expected value of k is undefined. By combining (3.3), (3.4), (3.5), and (3.6) we obtain

$$E(k) = 1 + \sum_{u=1}^{\infty} \int_{f_1(u)}^{\infty} g_1(z)\, dz - \lim_{N \to \infty} \frac{c_1 N}{[f_1(N)]^{t-1}}.$$

Thus the proof of the theorem is complete.

LEMMA. If

$$\lim_{N \to \infty} \frac{N}{[f_1(N)]^{t-1}}$$

exists, then the expected value of k is bounded.

The lemma is proved by showing that

$$\sum_{u=1}^{\infty} \int_{f_1(u)}^{\infty} g_1(z)\, dz$$

exists. From elementary integral calculus, we may write

$$\sum_{u=1}^{\infty} \int_{f_1(u)}^{\infty} g_1(z)\, dz = \sum_{u=1}^{N-1} \int_{f_1(u)}^{\infty} g_1(z)\, dz + \sum_{u=N}^{\infty} \int_{f_1(u)}^{\infty} g_1(z)\, dz.$$

Because $g_1(z)$ is a density function and we are summing over a finite number of terms,

$$\sum_{u=1}^{N-1} \int_{f_1(u)}^{\infty} g_1(z)\, dz$$

exists. By (3.1d) we may write

$$\sum_{u=N}^{\infty} \int_{f_1(u)}^{\infty} g_1(z)\, dz \leq \sum_{u=N}^{\infty} \int_{f_1(u)}^{\infty} \frac{1}{z^t}\, dz = c_1 \sum_{u=N}^{\infty} \frac{1}{[f_1(u)]^{t-1}}. \tag{3.7}$$

This expression exists because of Condition (3.1g). Hence the lemma is proved.

ILLUSTRATIONS. Although the given conditions appear to be rather restrictive, in several important cases verification can be readily made. Examples of two such cases follow:

Example 1. The expected value of k is obtained for the sample size necessary to insure a β^2 width coefficient and a $1 - \alpha$ confidence interval for the population mean, μ, when sampling from a normal population with mean μ and variance σ^2.

From Example 2 in Reference (5) we have

$$\frac{2t_{\alpha/2}(n)\sqrt{z}\sqrt{\chi^2_{1-\beta}(n)}}{\sqrt{\chi^2_{\beta}(m)}\sqrt{n(n-1)}} = d,$$

or

$$\frac{k(k-1)}{t^2_{\alpha/2}(k)\ \chi^2_{1-\beta}(k)} \geq \frac{4z}{\chi^2_{\beta}(m)\ d^2},$$

where k is the smallest integer $\geq n$.

Solving for z we obtain

$$z \leq \left[\frac{u(u-1)}{t^2_{\alpha/2}(u)\ \chi^2_{1-\beta}(u)}\right]\left[\frac{\chi^2_{\beta}(m)\ d^2}{4}\right] = f_1(u).$$

We observe that z is a monotonically increasing function in u for all u $> k$. Since z/σ^2 is distributed as Chi square, we may write

$$g_1\left(\frac{z}{\sigma^2}\right) = \frac{(z/\sigma^2)^{(p-2)/2}\ e^{-(z/2\sigma^2)}}{2^{p/2}\ [(p-2)/2]!},$$

where $p = m - 1$. Also $g_1(z)$ is monotonically decreasing for $z > z^*$, where z^* is the mode of the density function of z, and

$$\lim_{z\to\infty} \frac{g_1(z)}{(1/z)^t} = 0.$$

N is a value of u such that

$$g_1[f_1(N)] < \frac{1}{[f_1(N)]^t} \qquad \text{for } t = 3 \text{ for all } n \geq N.$$

The existence of N may be quite readily explained by determining for what values of z is $g_1(z) < (1/z)^t$ or

$$\frac{z^{(p-2)/2}\, e^{-z/2}}{2^{p/2}\,[(p-2)/2]!} < \frac{1}{z^t}.$$

From this stipulated inequality we must have

$$z^{(p-2+2t)/2}\, e^{-z/2} < 2^{p/2}\, (\frac{p-2}{2})!\ .$$

Squaring both sides of this inequality we obtain

$$z^{p-2+2t}\, e^{-z} < 2^p\, [(\frac{p-2}{2})!]^2\ ,$$

or

$$(p - 2 + 2t) \log z < p \log 2 + 2 \log (\frac{p-2}{2})!\ .$$

When t = 3 we have

$$(p + 4) \log z - z < p \log 2 + 2 \log (\frac{p-2}{2})!\ .$$

Since there exists a z = z' such that the terms on the left side of the inequality are < 0 for all z > z', there must exist a $z = z^*$ such that the inequality holds for all $z > z^*$. We shall show that

$$\sum_{u=N}^{\infty} \frac{1}{[f_1(u)]^2}$$

converges. We prove convergence by the comparison test. We need to show

$$\sum_{u=N}^{\infty} \frac{1}{[f_1(u)]^2} = \left[\frac{\chi^2_\beta(m)\ d^2}{4}\right]^2 \sum_{u=N}^{\infty} \frac{[t^2_{\alpha/2}(u)\ \chi^2_{1-\beta}(u)]^2}{[u(u-1)]^2}$$

$$< \left[\frac{\chi^2_\beta(m)\ d^2}{4}\right]^2 \left[\frac{\chi^2_{1-\beta}(N)}{N}\right]^2 [t^2_{\alpha/2}(N)]^2 \sum_{u=N}^{\infty} \frac{1}{(u-1)^2}.$$

Because $[\chi^2_{1-\beta}(n)]/n$ is monotonically decreasing and > 1 for all n, then

$$\left[\frac{\chi^2_{1-\beta}(N)}{N}\right]^2 > \left[\frac{\chi^2_{1-\beta}(u)}{u}\right]^2$$

for all $u > N$. Because $t_{\alpha/2}(n)$ is monotonically decreasing, we may write

$$\left[t^2_{\alpha/2}(u)\right]^2 < \left[t^2_{\alpha/2}(N)\right]^2$$

for all $u > N$. Since

$$\sum_{u=N}^{\infty} \frac{1}{(u - 1)^2}$$

converges, the proof is complete. Thus we see that Conditions (3.1a) to (3.1h) are satisfied.

We also have

$$\lim_{N\to\infty} \frac{N}{[f_1(N)]^2} = c_2 \lim_{N\to\infty} N \frac{[t^2_{\alpha/2}(N)\ \chi^2_{1-\beta}(N)]^2}{N^2(N - 1)^2}$$

$$= c_2 \lim_{N\to\infty} \frac{t^2_{\alpha/2}(N)}{\sqrt{N}} \lim_{N\to\infty} \frac{\chi^2_{1-\beta}(N)}{N - 1}$$

$$= 0,$$

where c_2 is a constant.

Making use of the lemma we may write

$$E(k) = 1 + \sum_{u=1}^{\infty} \int_{(1/\sigma^2)f_1(u)}^{\infty} \left[g_1\left(\frac{z}{\sigma^2}\right) d\left(\frac{z}{\sigma^2}\right)\right],$$

where

$$f_1(u) = \frac{u(u - 1)\ \chi^2_{\beta}(m)\ d^2}{4t^2_{\alpha/2}(u)\ \chi^2_{1-\beta}(u)} .$$

Example 2. The expected value of k is obtained for the necessary sample size to insure a β^2 width coefficient and a $1 - \alpha$ confidence interval on σ^2 when sampling from a normal population with mean α and variance σ^2.

From Example 1 in Reference (5)

$$\chi^2_{1-\beta}(n)\,\frac{G(n)z}{\chi^2_{\beta}(m)} = d,$$

or

$$\chi^2_{1-\beta}(k)\,\frac{G(k)z}{\chi^2_{\beta}(m)} \leq d,$$

where

$$G(n) = \frac{1}{\chi^2_{1-\alpha/2}(n)} - \frac{1}{\chi^2_{\alpha/2}(n)},$$

and k is the smallest value of $u > n$ where u is an integer.

Solving for z we obtain

$$z = \frac{d\,\chi^2_{\beta}(m)}{\chi^2_{1-\beta}(u)G(u)} = f_1(u).$$

We observe z is a monotonically increasing function in u. Since z/σ^2 is distributed at the Chi-square distribution with $(m - 1) = p$ degrees of freedom, we may write

$$g_1\left(\frac{z}{\sigma^2}\right) = \frac{(z/\sigma^2)^{(p-2)/2}\,e^{-(z/2\sigma^2)}}{2^{p/2}[(p - 2)/2]!}\,.$$

Also $g_1(z)$ is monotonically decreasing for $z > z^*$ where z^* is the mode of the density of z and $g_1(z) = O(1/z)^t$ for all $t > 0$. N is a value of n such that $g_1[f_1(N)] < 1/[f_1(N)]^t$ for $t > 0$ and $n \geq N$. We shall show

$$\sum_{u=N}^{\infty} \frac{1}{[f_1(u)]^t} \tag{3.8}$$

converges where $t = 4$. Convergence can be proved by substituting Fisher's approximation (6) for the Chi-square deviates in $f_1(u)$ and examining the summation of terms involving u for convergence. We obtain

$$\sum_{u=N}^{\infty}\left\{(\sqrt{2u-1} - v_{1-\beta})^2 \left[\frac{1}{(\sqrt{2u-1} \quad v_{1-\alpha/2})^2} - \frac{1}{(\sqrt{2u-1} + v_{1-\alpha/2})^2}\right]\right\}^4$$

$$= \sum_{u=N}^{\infty}\left\{ (\sqrt{2u-1} - v_{1-\beta})^2 \left[\frac{4\sqrt{2u-1}\, v_{1-\alpha/2}}{(2u-1 - v^2_{1-\alpha/2})^2}\right]\right\}^4 , \tag{3.9}$$

where v_γ is such that

$$\int_{v_\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}\, dx = \gamma.$$

To prove that (3.9) converges, we shall use Gauss's test. If $a_n > 0$ and

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\sigma}{n} + O\left(\frac{1}{n^{1+\lambda}}\right); \ \lambda > 0.$$

Then $\Sigma\, a_n$ converges or properly diverges according as $\sigma > 1$ or $\sigma < 1$.

Application of Gauss's test yields:

$$\left\{\left[\frac{4(\sqrt{2u-1} - v_{1-\beta})^2 \sqrt{2u-1}\, v_{1-\alpha/2}}{(2u - 1 - v^2_{1-\alpha/2})^2}\right]\left[\frac{(2u + 1 - v^2_{1-\alpha/2})^2}{4(\sqrt{2u+1} - v_{1-\beta})^2 \sqrt{2u+1}\, v_{1-\alpha/2}}\right]\right\}^4$$

$$= \left\{\left[\frac{2u + 1 - v^2_{1-\alpha/2}}{2u - 1 - v^2_{1-\alpha/2}}\right]^4 \left(\frac{2u - 1}{2u + 1}\right)^3 \left[\frac{1 - v_{1-\beta}/\sqrt{2u-1}}{1 - v_{1-\beta}/\sqrt{2u+1}}\right]^4\right\}^2 \quad (=)$$

$$= \left[1 + \frac{1}{u} + 0\left(\frac{1}{u^2}\right)\right]^4 \left[1 - \frac{1}{u} + 0\left(\frac{1}{u^2}\right)\right]^3 \left[1 - \frac{v_{1-\beta}}{2\sqrt{2}\,u^{3/2}} + 0\left(\frac{1}{u^2}\right)\right]^4$$

$$= \left[1 + \frac{1}{u} + 0\left(\frac{1}{u^2}\right)\right]^2 \left[1 - \frac{v_{1-\beta}}{2\sqrt{2}\,u^{3/2}} + 0\left(\frac{1}{u^2}\right)\right]$$

$$= 1 + \frac{2}{u} + 0\left(\frac{1}{u^{3/2}}\right).$$

Hence $\sigma = 2$, and the convergence of (3.9) is proved.

In order to show the convergence of (3.8), we shall use the Limit Comparison Test. We shall compare the terms of (3.8) with those of (3.9). Suppose we denote the terms of Fisher's approximation of the sum by b_i. Let a_i denote the terms of the true values of the series. We shall show by the comparison test that $\sum_{i=N}^{\infty} a_i$ converges. If $a_n > 0$ and $b_n > 0$ for $n = 1,2 \cdots$ and $\lim_{n\to\infty} (a_n/b_n) = c$, where c is a constant, then if $\sum_{i=N}^{\infty} b_i$ converges, we may conclude $\sum_{i=N}^{\infty} a_i$ also converges. In order to show $\sum_{u=N}^{\infty} 1/[f_1(u)]^t$ converges, we shall show that the limit of a_n/b_n is a constant as n tends to infinity.

$$\lim_{u\to\infty} \frac{\chi^2_{1-\beta}(u)\left[\dfrac{\chi^2_{\alpha/2}(u) - \chi^2_{1-\alpha/2}(u)}{\chi^2_{\alpha/2}(u)\,\chi^2_{1-\alpha/2}(u)}\right]}{(\sqrt{2u-1} - v_{1-\beta})^2\left[\dfrac{(\sqrt{2u-1} + v_{\alpha/2})^2 - (\sqrt{2u-1} - v_{\alpha/2})^2}{(2u - 1 - v^2_{1-\alpha/2})^2}\right]}$$

$$= \lim_{u\to\infty} \frac{\chi^2_{1-\beta}(u)}{2u\left[\sqrt{1-\frac{1}{2u}} - \frac{v_{1-\beta}}{\sqrt{2u}}\right]^2} \lim_{u\to\infty} \frac{4u^2\left(1 - \frac{1}{2u} - \frac{v^2_{1-\alpha/2)}}{2u}\right)^2}{\chi^2_{\alpha/2}(u)\,\chi^2_{1-\alpha/2}(u)}$$

$$\times \lim_{u\to\infty} \frac{\chi^2_{\alpha/2}(u) - \chi^2_{1-\alpha/2}(u)}{(\sqrt{2u-1} + v_{\alpha/2})^2 - (\sqrt{2u-1} - v_{\alpha/2})^2} \quad (=)$$

$$= \frac{1}{2} \times 4 \times \lim_{u\to\infty} \left[\frac{\chi^2_{\alpha/2}(u) - \chi^2_{1-\alpha/2}(u)}{4\sqrt{2-1/u} \quad v_{\alpha/2} \sqrt{u}} \right].$$

Since the difference of two Chi-square fractiles increases in proportion with $\sqrt{u}$, page 295 of Reference (7), the above limit exists.

Thus we have established that the given conditions of Theorem (3.1) are satisfied.

We shall proceed by evaluating

$$\lim_{N\to\infty} c_1 \frac{N}{[f_1(N)]^4}.$$

This may be written as

$$c_1 \lim_{N\to\infty} N \left[\chi^2_{1-\beta}(N) \left(\frac{1}{\chi^2_{1-\alpha/2}(N)} - \frac{1}{\chi^2_{\alpha/2}(N)} \right) \right]^4$$

$$= c_1 \lim_{N\to\infty} \left[\frac{\chi^2_{1-\beta}(N)}{N} \right]^4 \lim_{N\to\infty} N \left[\frac{N}{\chi^2_{1-\alpha/2}(N)} - \frac{N}{\chi^2_{\alpha/2}(N)} \right]^4$$

$$= \lim_{N\to\infty} N^{5/4} \left[\frac{1}{\chi^2_{1-\alpha/2}(N)} - \frac{1}{\chi^2_{\alpha/2}(N)} \right]. \qquad (3.10)$$

Substituting Fisher's approximation, (6), we obtain

$$c_1 \lim_{N\to\infty} N^{5/4} \left[\frac{1}{(\sqrt{2N-1} - v_{1-\alpha/2})^2} - \frac{1}{(\sqrt{2N-1} + v_{1-\alpha/2})^2} \right]$$

$$= c_1 \lim_{N\to\infty} N^{5/4} \frac{4\sqrt{2N-1}\ v_{1-\alpha/2}}{(2N-1 - v^2_{1-\alpha/2})^2}$$

$$= c_1 \lim_{N\to\infty} \frac{N^{5/4}\ 4N^{1/2} \sqrt{2-1/N}\ v_{1-\alpha/2}}{N^2[2 - (1/N) - (v^2_{1-\alpha/2})/N]^2} \ (=)$$

$$= c_1 \lim_{N\to\infty} \frac{4\sqrt{2-1/N}\, v_{1-\alpha/2}}{N^{1/4}(2 - 1/N - v^2_{1-\alpha/2}/N)^2}$$

$$= 0. \tag{3.11}$$

We shall show that (3.10) = (3.11). Since the difference between Fisher's approximation and the true value is of a smaller order than $1/N^{1/2}$ for all N greater than some specific value of N, we may write

$$\lim_{N\to\infty} N^{1/2}\left[\chi^2_\gamma(N) - (\sqrt{2N-1} - v_\gamma)^2\right] = 0.$$

We may also write

$$\lim_{N\to\infty} \left\{ N^{-3/4}\left[\chi^2_{1-\alpha/2}(N) - (\sqrt{2N-1} - v_{1-\alpha/2})^2 - \chi^2_{\alpha/2}(N) + (\sqrt{2N-1} + v_{1-\alpha/2})^2\right]\right\} = 0$$

or

$$\lim_{N\to\infty} \left\{ N^{-3/4}\left[(\sqrt{2N-1} + v_{1-\alpha/2})^2 - (\sqrt{2N-1} - v_{1-\alpha/2})^2\right]\right.$$

$$\left. - N^{-3/4}\left[\chi^2_{\alpha/2}(N) - \chi^2_{1-\alpha/2}(N)\right]\right\} = 0.$$

This may be written

$$\lim_{N\to\infty} \left\{ N^{-3/4}\, N^2\left[\frac{(\sqrt{2N-1} + v_{1-\alpha/2})^2\,(\sqrt{2N-1} - v_{1-\alpha/2})^2}{N^2}\right]\right.$$

$$\left. \times \left[\frac{1}{(\sqrt{2N-1} - v_{1-\alpha/2})^2} - \frac{1}{(\sqrt{2N-1} + v_{1-\alpha/2})^2}\right]\right\}$$

$$= \lim_{N\to\infty} \left\{ N^{-3/4}\, N^2\left[\frac{\chi^2_{1-\alpha/2}(N)\,\chi^2_{\alpha/2}(N)}{N^2}\right]\left[\frac{1}{\chi^2_{1-\alpha/2}(N)} - \frac{1}{\chi^2_{\alpha/2}(N)}\right]\right\};$$

or

$$\lim_{N\to\infty}\left[\frac{(\sqrt{2N-1}+v_{1-\alpha/2})^2\,(\sqrt{2N-1}-v_{1-\alpha/2})^2}{N^2}\right]$$

$$\times \lim_{N\to\infty} N^{5/4}\left[\frac{1}{(\sqrt{2N-1}-v_{1-\alpha/2})^2}-\frac{1}{(\sqrt{2N-1}+v_{1-\alpha/2})^2}\right]$$

$$= \lim_{N\to\infty}\left[\frac{\chi^2_{1-\alpha/2}(N)\chi^2_{\alpha/2}(N)}{N^2}\right]\lim_{N\to\infty} N^{5/4}\left[\frac{1}{\chi^2_{1-\alpha/2}(N)}-\frac{1}{\chi^2_{\alpha/2}(N)}\right].$$

Thus we have shown that when Fisher's approximation is substituted for the true Chi-square deviate, we may write

$$\lim_{N\to\infty} N^{5/4}\left[\frac{1}{(\sqrt{2N-1}-v_{1-\alpha/2})^2}-\frac{1}{(\sqrt{2N-1}+v_{1-\alpha/2})^2}\right]$$

$$= \lim_{N\to\infty} N^{5/4}\left[\frac{1}{\chi^2_{1-\alpha/2}(N)}-\frac{1}{\chi^2_{\alpha/2}(N)}\right].$$

We have shown the left side of this equality to be 0. Therefore, we have established

$$E(k) = 1 + \sum_{u=1}^{\infty}\int_{[f_1(u)]/\sigma^2}^{\infty} g_1(z)\,dz,$$

where

$$f_1(u) = \frac{d\,\chi^2_{\beta}(m)}{\chi^2_{1-\beta}(u)G(u)}\,.$$

The tabular results appear in Chapter VIII.

CHAPTER IV

NECESSARY SAMPLE SIZE FOR THE DESIRED WIDTH INTERVAL ON THE RATIO OF VARIANCES

We shall examine the problem of obtaining sample sizes when making an interval estimate of the ratio of variances from two independent normal populations. Using Graybill's technique (5), we will determine how large the sample sizes should be so that on the ratio of σ_2^2/σ_1^2: (1) the confidence coefficient, the probability the width includes the true ratio, is a specified $1 - \alpha$, and (2) the width coefficient, the probability the width $<$ d specified units, is greater than or equal to a specified β^2.

We shall present a two-step procedure originated by Graybill (5).

Suppose z_{1i} is distributed normally with mean μ_1 and variance σ_1^2, and z_{2j} is distributed normally with mean μ_2 and variance σ_2^2. Let z_{1i} and z_{2j} be mutually independent.

A $1 - \alpha$ confidence interval on σ_2^2/σ_1^2 is given by

$$P\left[\frac{s_2^2}{s_1^2} F_{1-\alpha/2}(n_1, n_2) < \frac{\sigma_2^2}{\sigma_1^2} < \frac{s_2^2}{s_1^2} F_{\alpha/2}(n_1, n_2)\right] = 1 - \alpha,$$

where

$$s_1^2 = \sum_{i=1}^{n_1} \frac{(z_{1i} - \bar{z}_1)^2}{n_1 - 1}; \qquad s_2^2 = \sum_{j=1}^{n_2} \frac{(z_{2j} - \bar{z}_2)^2}{n_2 - 1};$$

$$\bar{z}_1 = \sum_{i=1}^{n_1} \frac{z_{1i}}{n_1}; \qquad \bar{z}_2 = \sum_{j=1}^{n_2} \frac{z_{2j}}{n_2};$$

n_1 is the size of the sample drawn from the z_{1i} population;

n_2 is the size of the sample drawn from the z_{2j} population;

and

$F_\gamma(n_1,n_2)$ is such that

$$\int_{F_\gamma(n_1,n_2)}^{\infty} W(F;n_1,n_2)\, dF = \gamma,$$

where

$W(F;n_1,n_2)$ follows the F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

Let w denote the width of the confidence interval so that

$$w = \frac{s_2^2}{s_1^2}\left[F_{\alpha/2}(n_1,n_2) - \frac{1}{F_{\alpha/2}(n_2,n_1)}\right].$$

Let

$$C(n_1,n_2) = F_{\alpha/2}(n_1,n_2) - \frac{1}{F_{\alpha/2}(n_2,n_1)};$$
$$\theta = \sigma_2^2/\sigma_1^2;$$

and

$$Y(n_2,n_1) = g(w;\theta;n_1,n_2) = \frac{w}{\theta C(n_1,n_2)}.$$

We observe that $g(w;\theta;n_1,n_2)$ is distributed independently of θ as the central F density with $n_2 - 1$ and $n_1 - 1$ degrees of freedom. Hence

$$P[Y(n_2,n_1) \leq F_{1-\beta}(n_2,n_1)] = \beta,$$

where

$$\int_0^{F_{1-\beta}(n_2,n_1)} W(F;n_2,n_1)\, dF = \beta.$$

Let $Y(n_2,n_1) = F_{1-\beta}(n_2,n_1)$. Solving for w yields

$$w = C(n_1,n_2)\ \theta F_{1-\beta}(n_2,n_1) = h(\theta;n_1,n_2).$$

If $n_2 \geq 3$, $n_1 \geq 3$, $\alpha \leq 0.20$, and $\beta \geq 0.80$, then

$$F_{1-\beta}(n_2,n_1) > F_{1-\beta}(n_2 + 1,n_1 + 1),$$

$$F_{\alpha/2}(n_1,n_2) > F_{\alpha/2}(n_1 + 1,n_2 + 1),$$

and

$$F_{\alpha/2}(n_2,n_1) > F_{\alpha/2}(n_2 + 1,n_1 + 1).$$

Thus we may write

$$\frac{C(n_1 + 1,n_2 + 1)\ F_{1-\beta}(n_2 + 1,n_1 + 1)}{C(n_1,n_2)\ F_{1-\beta}(n_2,n_1)}$$

$$= \frac{\{F_{\alpha/2}(n_1 + 1,n_2 + 1) - 1/[F_{\alpha/2}(n_2 + 1,n_1 + 1)]\}\ F_{1-\beta}(n_2 + 1,n_1 + 1)}{\{F_{\alpha/2}(n_1,n_2) - 1/[F_{\alpha/2}(n_2,n_1)]\}F_{1-\beta}(n_2,n_1)} < 1.$$

Therefore, we reach the following conclusions:

$h(\theta;n_1,n_2)$ is monotonically increasing in θ for every n_1 and n_2; (4.1a)

$h(\theta;n_1,n_2)$ is monotonically decreasing in n_1 and n_2 for every θ. (4.1b)

The first step in our procedure requires samples of size m_1 and m_2 and the following computations:

$$z = \sum_{q=1}^{m_2} \frac{(z_{2q} - \bar{z}_2^*)^2}{(m_2 - 1)} \frac{(m_1 - 1)}{\sum_{p=1}^{m_1}(z_{1p} - \bar{z}_1^*)^2},$$

where

$$\bar{z}_1^* = \sum_{p=1}^{m_1} \frac{z_{1p}}{m_1},$$

and

$$\bar{z}_2^* = \sum_{q=1}^{m_2} \frac{z_{2q}}{m_2}.$$

Then z/θ is distributed as the F distribution with $m_2 - 1$ and $m_1 - 1$ degrees of freedom. Hence

$$P\left[\frac{z}{\theta} > F_\beta(m_2,m_1)\right] = \beta,$$

or

$$P\left[\frac{z}{F_\beta(m_2,m_1)} > \theta\right] = \beta.$$

Let $t(z) = z/F_\beta(m_2,m_1)$. If we consider the joint density of $t(z)$ and w, we may write

$$P(w \leq d) \geq P[w \leq d,\ t(z) > \theta] = P[w \leq d | t(z) > \theta]\ P[t(z) > \theta]. \quad (4.2)$$

Since $g(w;\theta;n_1,n_2)$ follows the F distribution, then

$$P\left[\frac{w}{\theta C(n_1,n_2)} < F_{1-\beta}(n_2,n_1)\right] = \beta,$$

or

$$P[w < h(\theta;n_1,n_2)] = \beta.$$

If we consider $\theta_1 > \theta$, then from (4.1a) we have

$$P[w < h(\theta_1;n_1,n_2) | \theta_1 \geq \theta] \geq P[w < h(\theta;n_1,n_2)].$$

When $t(z) > \theta$, then

$$P\{w < h[t(z);n_1,n_2] | t(z) > \theta\} \geq P\{w < h[t(z);n_1,n_2] | t(z) = \theta\} = \beta.$$

Since

$$P[t(z) > \theta] = \beta, \quad (4.3)$$

and if we set

$$h[t(z);n_1,n_2] = d, \quad (4.4)$$

then by combining (4.2), (4.3), and (4.4) we obtain

$$P(w < d) \geq \beta^2.$$

Thus, we choose the smallest integral values of n_1 and n_2 to satisfy

$$h[t(z); n_1, n_2] \leq d,$$

or

$$Q(n_1, n_2) = F_{1-\beta}(n_2, n_1)\ C(n_1, n_2) \leq \frac{dF_\beta(m_2, m_1)}{z}. \tag{4.5}$$

Table II provides values of $n_1^* = n_1 - 1$ and $n_2^* = n_2 - 1$ which satisfy (4.5). The solution is not unique, and there is little difficulty in minimizing $n_1^* + n_2^*$. (A practical example of this would be that if there were a cost of c_1 dollars per sample when taking n_1 samples and a cost of c_2 dollars per sample when taking n_2 samples, we would want to minimize the total cost C, where $C = c_1 n_1 + c_2 n_2$.)

We shall present some examples which illustrate the use of Table II and the optimal choices of n_1 and n_2.

Example 1. An experimenter desires a 95 per cent confidence interval on the ratio of variances with a width coefficient of 90 per cent. We have the following information available:

$m_1 = 14$	$\beta = 0.90$
$m_2 = 8$	$z = 1.115$
$d = 2$	$F_\beta(m_2, m_1) = 2.23$
$\alpha = 0.05$	$Q(n_1, n_2) = \frac{2 \times 2.23}{1.115} = 4$

The following is a list of possible results which are obtained from Table II. For some of these values, linear interpolation in Table II was used.

(Text continues on page 33.)

TABLE II

TABLES FOR FINDING SAMPLE SIZES NECESSARY FOR A SPECIFIED WIDTH CONFIDENCE INTERVAL ON THE RATIO OF VARIANCES FROM TWO INDEPENDENT NORMAL POPULATIONS

$1 - \alpha = 0.99,\ \beta = 0.90$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	255.75	129.43	87.66	68.10	57.22	50.28	45.57	42.14	39.55	37.62	35.98	34.62
4	193.41	95.01	62.84	48.06	39.81	34.61	31.18	28.59	26.71	25.24	24.05	23.09
5	164.09	78.83	51.31	38.73	31.85	27.48	24.56	22.43	20.81	19.60	18.63	17.82
6	147.27	69.63	44.85	33.49	27.29	23.41	20.82	18.94	17.52	16.41	15.55	14.86
7	136.12	63.70	40.59	30.23	24.40	20.83	18.39	16.69	15.38	14.41	13.59	12.95
8	128.56	59.67	37.78	27.89	22.40	19.08	16.78	15.19	13.96	13.02	12.24	11.64
9	122.98	56.53	35.59	26.14	21.00	17.76	15.58	14.07	12.92	11.99	11.27	10.71
10	118.94	54.38	33.96	24.84	19.81	16.77	14.69	13.18	12.07	11.20	10.49	9.94
11	115.36	52.49	32.64	23.82	18.95	15.93	13.91	12.52	11.44	10.61	9.91	9.40
12	112.89	50.96	31.58	22.97	18.24	15.28	13.34	11.97	10.94	10.12	9.44	8.90
13	110.43	49.67	30.83	22.27	17.64	14.83	12.85	11.52	10.49	9.71	9.05	8.52
14	108.39	48.62	30.10	21.71	17.15	14.33	12.46	11.16	10.10	9.32	8.69	8.22
15	106.86	47.83	29.43	21.24	16.77	13.98	12.14	10.80	9.83	9.06	8.44	7.93
16	105.36	47.03	28.87	20.83	16.44	13.67	11.85	10.53	9.57	8.81	8.20	7.70
17	104.33	46.45	28.47	20.43	16.07	13.37	11.58	10.28	9.33	8.58	7.98	7.49
18	103.32	45.92	28.09	20.12	15.82	13.15	11.32	10.08	9.10	8.36	7.77	7.32
19	102.34	45.42	27.74	19.84	15.57	12.93	11.11	9.89	8.94	8.20	7.62	7.13
20	101.41	44.95	27.41	19.58	15.33	12.73	10.93	9.72	8.77	8.05	7.47	6.99
25	98.41	43.17	26.18	18.60	14.51	12.00	10.27	9.10	8.19	7.49	6.92	6.47
30	96.40	42.10	25.47	18.03	14.02	11.51	9.86	8.68	7.77	7.11	6.56	6.11
40	93.90	40.80	24.56	17.31	13.34	10.96	9.32	8.18	7.30	6.65	6.12	5.69
50	92.43	40.03	24.03	16.89	12.98	10.64	9.02	7.90	7.05	6.41	5.89	5.46
60	91.43	39.51	23.66	16.52	12.75	10.37	8.84	7.72	6.86	6.23	5.73	5.31
120	88.97	38.24	22.78	15.83	12.15	9.84	8.30	7.23	6.41	5.79	5.30	4.89

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	33.58	32.58	31.85	31.11	30.54	30.03	28.16	26.97	25.57	24.75	24.24	23.01
4	22.26	21.58	20.99	20.48	20.05	19.66	18.35	17.46	16.41	15.88	15.50	14.62
5	17.15	16.58	16.08	15.72	15.32	15.03	13.88	13.16	12.36	11.87	11.55	10.82
6	14.26	13.75	13.33	12.99	12.66	12.40	11.40	10.76	10.01	9.61	9.33	8.71
7	12.43	11.96	11.57	11.21	10.91	10.70	9.83	9.23	8.57	8.17	7.91	7.32
8	11.13	10.71	10.35	10.04	9.75	9.53	8.70	8.15	7.53	7.19	6.95	6.41
9	10.24	9.82	9.47	9.17	8.90	8.71	7.90	7.36	6.78	6.45	6.24	5.69
10	9.49	9.11	8.78	8.49	8.23	8.05	7.29	6.79	6.22	5.91	5.68	5.18
11	8.97	8.60	8.27	8.00	7.74	7.53	6.80	6.31	5.76	5.46	5.27	4.76
12	8.48	8.12	7.81	7.54	7.29	7.13	6.38	5.93	5.40	5.08	4.89	4.42
13	8.11	7.76	7.45	7.19	6.97	6.79	6.06	5.61	5.09	4.78	4.60	4.16
14	7.81	7.46	7.16	6.90	6.69	6.48	5.78	5.35	4.85	4.54	4.37	3.91
15	7.53	7.20	6.90	6.65	6.44	6.23	5.55	5.12	4.63	4.33	4.15	3.71
16	7.30	6.97	6.70	6.45	6.22	6.03	5.35	4.93	4.43	4.14	3.97	3.52
17	7.10	6.77	6.50	6.26	6.03	5.85	5.17	4.75	4.27	3.98	3.81	3.37
18	6.95	6.63	6.34	6.10	5.88	5.69	5.01	4.60	4.11	3.83	3.66	3.24
19	6.75	6.43	6.17	5.93	5.71	5.53	4.87	4.47	3.99	3.71	3.55	3.11
20	6.61	6.30	6.04	5.80	5.58	5.40	4.75	4.35	3.88	3.60	3.44	2.99
25	6.12	5.81	5.54	5.32	5.12	4.93	4.29	3.91	3.44	3.24	3.06	2.59
30	5.78	5.48	5.21	4.99	4.80	4.62	4.00	3.62	3.14	2.89	2.74	2.32
40	5.35	5.07	4.82	4.60	4.42	4.24	3.64	3.24	2.82	2.53	2.38	1.97
50	5.12	4.81	4.58	4.36	4.17	4.01	3.41	3.03	2.60	2.34	2.18	1.77
60	4.95	4.64	4.41	4.20	4.01	3.85	3.27	2.91	2.45	2.19	2.03	1.62
120	4.54	4.24	4.02	3.82	3.63	3.48	2.90	2.53	2.09	1.86	1.66	1.25

TABLE II (continued)

$1 - \alpha = 0.99,\ \beta = 0.95$

n_2^*	n_1^* = 3	4	5	6	7	8	9	10	11	12	13	14
3	440.33	221.05	148.74	115.31	96.41	84.66	76.62	70.74	66.37	62.99	60.23	57.98
4	304.19	147.72	97.13	73.82	60.92	52.93	47.48	43.46	40.51	38.25	36.41	34.93
5	245.23	116.23	75.11	56.39	46.11	39.66	35.36	32.22	29.83	28.06	26.63	25.45
6	213.07	99.19	63.32	46.99	38.17	32.60	28.84	26.16	24.19	22.64	21.41	20.43
7	192.87	88.66	55.96	41.33	33.27	28.25	24.89	22.44	20.66	19.27	18.15	17.27
8	179.19	81.55	51.07	37.39	29.92	25.34	22.22	19.98	18.33	17.08	16.04	15.23
9	168.93	76.28	47.45	34.54	27.52	23.23	20.31	18.20	16.68	15.47	14.52	13.76
10	161.63	72.50	44.87	32.51	25.81	21.63	18.88	16.87	15.42	14.30	13.37	12.64
11	155.69	69.43	42.64	30.80	24.38	20.43	17.78	15.92	14.47	13.39	12.49	11.82
12	150.95	66.98	41.10	29.57	23.36	19.45	16.91	15.08	13.71	12.66	11.79	11.14
13	147.09	65.00	39.61	28.52	22.46	18.68	16.18	14.37	13.01	12.02	11.18	10.55
14	143.65	63.27	38.57	27.62	21.69	17.99	15.58	13.81	12.49	11.50	10.70	10.09
15	141.19	62.02	37.59	26.82	20.96	17.41	15.05	13.37	12.10	11.12	10.34	9.74
16	138.77	60.75	36.73	26.18	20.53	16.94	14.61	12.91	11.66	10.71	9.95	9.36
17	136.82	59.52	36.03	25.65	20.04	16.56	14.26	12.59	11.36	10.42	9.67	9.09
18	134.91	58.76	35.37	25.12	19.62	16.18	13.92	12.27	11.06	10.13	9.39	8.82
19	133.47	58.03	34.86	24.72	19.28	15.87	13.64	12.01	10.83	9.92	9.19	8.62
20	132.09	57.33	34.39	24.35	18.94	15.60	13.39	11.77	10.61	9.71	8.98	8.43
25	126.83	54.66	32.56	22.93	17.75	14.54	12.39	10.90	9.79	8.88	8.19	7.66
30	123.46	52.92	31.43	22.04	16.99	13.89	11.78	10.30	9.21	8.39	7.72	7.21
40	119.59	50.95	30.08	20.99	16.05	13.06	11.04	9.62	8.61	7.78	7.14	6.64
50	117.22	49.75	29.28	20.36	15.52	12.59	10.61	9.22	8.21	7.44	6.81	6.33
60	115.75	48.81	28.76	19.88	15.20	12.31	10.36	8.99	7.97	7.21	6.61	6.10
120	112.36	47.08	27.45	18.96	14.35	11.55	9.69	8.37	7.40	6.65	6.08	5.58

n_2^*	n_1^* = 15	16	17	18	19	20	25	30	40	50	60	120
3	56.17	54.54	53.27	52.08	51.06	50.20	46.97	44.97	42.62	41.24	40.33	38.27
4	33.64	32.65	31.78	30.97	30.33	29.70	27.64	26.23	24.66	23.82	23.27	21.90
5	24.46	23.61	22.92	22.36	21.81	21.35	19.71	18.68	17.44	16.74	16.30	15.26
6	19.57	18.85	18.24	17.72	17.29	16.90	15.53	14.64	13.58	13.02	12.64	11.76
7	16.54	15.93	15.38	14.92	14.53	14.21	12.99	12.19	11.27	10.77	10.41	9.59
8	14.52	13.99	13.49	13.09	12.73	12.41	11.28	10.55	9.73	9.27	8.93	8.18
9	13.12	12.55	12.08	11.71	11.38	11.10	10.05	9.36	8.57	8.14	7.88	7.18
10	12.03	11.52	11.07	10.72	10.40	10.14	9.13	8.49	7.79	7.35	7.06	6.42
11	11.24	10.75	10.31	9.98	9.67	9.42	8.42	7.80	7.11	6.69	6.46	5.83
12	10.58	10.10	9.69	9.32	9.03	8.79	7.86	7.25	6.57	6.19	5.94	5.36
13	10.01	9.55	9.14	8.83	8.56	8.30	7.40	6.82	6.17	5.81	5.57	4.98
14	9.56	9.11	8.71	8.40	8.15	7.90	7.01	6.47	5.82	5.44	5.21	4.66
15	9.22	8.77	8.39	8.05	7.80	7.55	6.69	6.16	5.53	5.16	4.93	4.37
16	8.85	8.42	8.06	7.77	7.50	7.28	6.41	5.89	5.29	4.93	4.70	4.14
17	8.58	8.16	7.81	7.49	7.22	7.01	6.16	5.65	5.06	4.71	4.49	3.93
18	8.35	7.93	7.56	7.25	6.99	6.78	5.96	5.46	4.86	4.51	4.30	3.77
19	8.10	7.69	7.35	7.03	6.78	6.57	5.77	5.25	4.66	4.34	4.13	3.60
20	7.91	7.50	7.17	6.86	6.61	6.40	5.61	5.10	4.52	4.18	3.99	3.47
25	7.19	6.80	6.46	6.17	5.95	5.74	5.01	4.52	3.95	3.70	3.51	2.94
30	6.75	6.37	6.04	5.76	5.55	5.34	4.61	4.14	3.59	3.28	3.09	2.60
40	6.18	5.84	5.53	5.25	5.05	4.85	4.12	3.66	3.15	2.84	2.66	2.17
50	5.88	5.53	5.24	4.97	4.73	4.55	3.86	3.41	2.88	2.60	2.41	1.93
60	5.66	5.31	5.03	4.76	4.55	4.38	3.68	3.24	2.70	2.42	2.22	1.76
120	5.16	4.82	4.55	4.30	4.07	3.90	3.22	2.80	2.29	2.02	1.80	1.33

TABLE II (continued)

$1 - \alpha = 0.99, \beta = 0.99$

n_2^*	n_1^* = 3	4	5	6	7	8	9	10	11	12	13	14
3	1397.85	695.88	466.19	359.97	300.43	263.28	237.78	219.40	205.56	194.95	185.93	178.60
4	770.39	369.41	240.80	182.28	149.86	129.69	116.00	106.03	98.72	93.00	88.39	84.75
5	546.67	255.09	163.16	121.54	98.75	84.51	75.08	68.31	63.20	59.29	56.12	53.58
6	437.78	200.36	126.20	93.00	74.90	63.63	56.13	50.70	46.75	43.70	41.21	39.21
7	374.66	168.94	105.01	76.80	61.44	51.81	45.38	40.92	37.53	34.92	32.86	31.15
8	334.16	148.87	91.76	66.53	52.92	44.42	38.73	34.81	31.79	29.53	27.69	26.21
9	305.92	134.91	82.63	59.45	47.01	39.33	34.17	30.58	27.88	25.75	24.13	22.78
10	285.36	124.79	76.00	54.42	42.83	35.66	30.95	27.54	25.08	23.14	21.58	20.33
11	269.75	117.17	70.88	50.53	39.52	32.82	28.38	25.27	22.89	21.12	19.64	18.51
12	257.35	111.16	66.87	47.51	37.19	30.70	26.51	23.49	21.27	19.58	18.18	17.09
13	247.60	106.29	63.75	45.13	35.12	29.00	24.92	22.07	19.89	18.31	17.01	15.93
14	239.14	102.33	61.11	43.22	33.52	27.59	23.69	20.93	18.83	17.27	16.05	15.05
15	232.60	99.11	59.11	41.53	32.14	26.38	22.60	19.93	17.98	16.46	15.24	14.27
16	226.58	96.27	57.22	40.14	31.10	25.44	21.74	19.14	17.19	15.71	14.57	13.62
17	221.48	93.91	55.65	38.95	30.07	24.61	20.99	18.45	16.59	15.15	13.99	13.06
18	217.31	91.85	54.27	37.88	29.28	23.92	20.37	17.87	16.05	14.60	13.50	12.60
19	213.63	90.05	53.06	37.04	28.50	23.23	19.76	17.31	15.56	14.17	13.06	12.17
20	210.48	88.50	52.03	36.25	27.88	22.66	19.24	16.88	15.16	13.75	12.70	11.83
25	198.51	82.78	48.34	33.43	25.49	20.64	17.45	15.24	13.57	12.30	11.31	10.50
30	190.68	79.09	45.97	31.60	23.97	19.40	16.31	14.22	12.59	11.40	10.46	9.68
40	181.49	74.77	43.10	29.51	22.26	17.91	15.00	13.02	11.52	10.35	9.46	8.72
50	176.45	72.29	41.60	28.27	21.30	17.02	14.25	12.33	10.86	9.77	8.90	8.19
60	173.21	70.69	40.53	27.56	20.67	16.53	13.81	11.87	10.46	9.39	8.53	7.87
120	164.57	66.68	38.00	25.65	19.15	15.15	12.65	10.82	9.50	8.43	7.65	7.00

n_2^*	n_1^* = 15	16	17	18	19	20	25	30	40	50	60	120
3	173.49	168.40	164.40	160.68	157.56	154.71	144.50	138.24	130.90	126.65	123.78	117.37
4	81.61	79.11	76.91	74.91	73.35	71.79	66.63	63.20	59.35	57.21	55.81	52.46
5	51.46	49.69	48.13	46.92	45.72	44.72	41.12	38.94	36.33	34.83	33.85	31.63
6	37.56	36.15	34.98	33.95	33.06	32.26	29.48	27.79	25.72	24.61	23.87	22.19
7	29.81	28.62	27.66	26.78	26.03	25.41	23.04	21.57	19.91	18.98	18.35	16.88
8	24.97	23.96	23.07	22.34	21.68	21.11	19.07	17.80	16.31	15.48	14.93	13.69
9	21.70	20.73	19.96	19.30	18.69	18.18	16.32	15.18	13.86	13.11	12.66	11.49
10	19.32	18.47	17.75	17.14	16.57	16.14	14.42	13.36	12.17	11.48	10.99	9.96
11	17.56	16.76	16.09	15.51	14.98	14.57	12.96	11.96	10.85	10.17	9.81	8.78
12	16.19	15.43	14.79	14.24	13.74	13.35	11.82	10.91	9.80	9.19	8.84	7.90
13	15.11	14.38	13.77	13.24	12.79	12.39	10.92	10.05	9.02	8.44	8.09	7.22
14	14.22	13.51	12.92	12.41	11.99	11.60	10.22	9.36	8.36	7.80	7.46	6.63
15	13.46	12.74	12.21	11.72	11.34	10.89	9.60	8.76	7.80	7.26	6.96	6.14
16	12.84	12.17	11.65	11.18	10.74	10.37	9.08	8.30	7.37	6.85	6.53	5.73
17	12.30	11.65	11.15	10.68	10.26	9.93	8.64	7.88	7.00	6.47	6.17	5.38
18	11.88	11.25	10.72	10.26	9.85	9.50	8.30	7.53	6.64	6.15	5.85	5.11
19	11.44	10.85	10.33	9.88	9.51	9.17	7.93	7.21	6.37	5.86	5.57	4.83
20	11.11	10.49	10.02	9.58	9.18	8.88	7.66	6.92	6.10	5.61	5.34	4.60
25	9.85	9.28	8.81	8.40	8.05	7.74	6.64	5.98	5.18	4.83	4.55	3.78
30	9.07	8.52	8.07	7.68	7.35	7.05	6.00	5.35	4.59	4.17	3.93	3.28
40	8.12	7.63	7.20	6.83	6.52	6.25	5.26	4.63	3.94	3.51	3.27	2.67
50	7.61	7.11	6.72	6.37	6.05	5.77	4.82	4.24	3.53	3.15	2.91	2.33
60	7.26	6.78	6.40	6.06	5.74	5.50	4.57	3.99	3.28	2.90	2.67	2.08
120	6.46	6.00	5.62	5.29	5.00	4.77	3.87	3.32	2.67	2.33	2.08	1.50

TABLE II (continued)

$1 - \alpha = 0.95,\ \beta = 0.90$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	82.87	52.94	40.85	34.49	30.68	28.09	26.26	24.90	23.81	22.97	22.30	21.68
4	62.85	39.03	29.50	24.55	21.53	19.51	18.15	17.08	16.29	15.62	15.11	14.65
5	53.40	32.47	24.18	19.88	17.34	15.60	14.38	13.49	12.75	12.22	11.79	11.39
6	48.00	28.75	21.19	17.23	14.88	13.32	12.25	11.43	10.79	10.28	9.86	9.54
7	44.36	26.31	19.18	15.58	13.32	11.90	10.86	10.10	9.51	9.07	8.65	8.35
8	41.92	24.68	17.89	14.38	12.36	10.89	9.91	9.19	8.63	8.18	7.81	7.53
9	40.11	23.37	16.85	13.49	11.50	10.17	9.23	8.54	8.00	7.57	7.20	6.93
10	38.80	22.49	16.09	12.83	10.86	9.62	8.68	8.01	7.49	7.05	6.71	6.45
11	37.68	21.73	15.49	12.31	10.40	9.12	8.24	7.60	7.10	6.69	6.36	6.09
12	36.84	21.10	14.97	11.89	10.02	8.77	7.91	7.28	6.79	6.40	6.05	5.78
13	36.05	20.58	14.63	11.52	9.68	8.50	7.62	7.00	6.52	6.14	5.82	5.53
14	35.39	20.15	14.29	11.23	9.43	8.23	7.40	6.79	6.29	5.91	5.58	5.34
15	34.88	19.82	13.96	11.00	9.21	8.03	7.21	6.58	6.12	5.74	5.42	5.16
16	34.40	19.51	13.72	10.79	9.01	7.86	7.03	6.43	5.95	5.58	5.28	5.00
17	34.06	19.28	13.54	10.59	8.83	7.67	6.87	6.26	5.80	5.44	5.12	4.87
18	33.73	19.06	13.36	10.44	8.69	7.54	6.72	6.14	5.66	5.31	4.99	4.77
19	33.42	18.84	13.19	10.29	8.55	7.44	6.60	6.05	5.55	5.20	4.89	4.64
20	33.11	18.62	13.02	10.14	8.44	7.31	6.50	5.93	5.47	5.10	4.81	4.54
25	32.08	17.88	12.44	9.64	7.98	6.90	6.10	5.55	5.10	4.75	4.46	4.22
30	31.47	17.45	12.09	9.35	7.71	6.60	5.87	5.30	4.86	4.50	4.23	3.98
40	30.66	16.91	11.67	8.96	7.35	6.30	5.55	5.00	4.55	4.23	3.95	3.70
50	30.17	16.59	11.40	8.75	7.15	6.13	5.37	4.82	4.40	4.06	3.79	3.56
60	29.85	16.38	11.24	8.56	7.01	5.96	5.26	4.72	4.28	3.97	3.69	3.45
120	29.05	15.85	10.82	8.20	6.69	5.67	4.94	4.42	3.99	3.68	3.42	3.19

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	21.22	20.81	20.45	20.10	19.84	19.63	18.75	18.19	17.49	17.09	16.83	16.23
4	14.26	13.95	13.64	13.45	13.22	13.03	12.38	11.96	11.44	11.17	10.95	10.47
5	11.10	10.80	10.57	10.38	10.18	10.05	9.48	9.10	8.65	8.40	8.25	7.82
6	9.24	9.01	8.80	8.63	8.46	8.34	7.82	7.48	7.06	6.84	6.70	6.35
7	8.08	7.86	7.67	7.48	7.32	7.22	6.77	6.45	6.07	5.84	5.71	5.36
8	7.27	7.04	6.86	6.71	6.56	6.44	5.99	5.70	5.36	5.16	5.02	4.71
9	6.68	6.48	6.29	6.14	6.00	5.91	5.48	5.15	4.83	4.64	4.51	4.20
10	6.22	6.03	5.84	5.70	5.56	5.45	5.04	4.77	4.44	4.26	4.13	3.83
11	5.88	5.67	5.52	5.36	5.22	5.11	4.71	4.45	4.11	3.94	3.83	3.52
12	5.56	5.38	5.20	5.07	4.94	4.85	4.42	4.17	3.87	3.69	3.57	3.26
13	5.31	5.14	4.99	4.84	4.71	4.63	4.22	3.96	3.65	3.47	3.35	3.07
14	5.13	4.96	4.79	4.66	4.53	4.41	4.01	3.77	3.47	3.29	3.17	2.91
15	4.95	4.78	4.61	4.48	4.36	4.24	3.85	3.61	3.32	3.14	3.03	2.74
16	4.81	4.63	4.48	4.34	4.22	4.11	3.71	3.48	3.17	3.00	2.89	2.61
17	4.66	4.50	4.34	4.22	4.09	3.98	3.60	3.37	3.06	2.90	2.78	2.51
18	4.57	4.40	4.24	4.12	4.00	3.88	3.49	3.24	2.96	2.78	2.66	2.40
19	4.44	4.28	4.12	4.00	3.89	3.77	3.40	3.15	2.87	2.69	2.58	2.31
20	4.36	4.19	4.05	3.91	3.79	3.68	3.31	3.09	2.79	2.61	2.50	2.23
25	4.03	3.85	3.72	3.59	3.48	3.38	2.99	2.76	2.47	2.32	2.20	1.93
30	3.80	3.64	3.50	3.37	3.26	3.17	2.78	2.55	2.26	2.11	2.01	1.73
40	3.52	3.37	3.23	3.11	3.00	2.91	2.54	2.30	2.04	1.86	1.74	1.46
50	3.38	3.20	3.08	2.95	2.85	2.74	2.39	2.15	1.87	1.70	1.60	1.31
60	3.26	3.09	2.96	2.83	2.73	2.62	2.27	2.06	1.76	1.60	1.50	1.21
120	2.99	2.82	2.70	2.58	2.47	2.38	2.02	1.80	1.51	1.34	1.22	0.92

TABLE II (continued)

$1 - \alpha = 0.95,\ \beta = 0.95$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	142.68	90.41	69.31	58.40	51.70	47.30	44.15	41.80	39.95	38.46	37.32	36.32
4	98.85	60.68	45.59	37.71	32.95	29.83	27.65	25.97	24.71	23.67	22.88	22.16
5	79.80	47.88	35.40	28.94	25.10	22.52	20.70	19.38	18.27	17.49	16.85	16.26
6	69.44	40.95	29.91	24.17	20.82	18.56	16.97	15.78	14.89	14.18	13.58	13.11
7	62.86	36.62	26.44	21.30	18.15	16.14	14.69	13.58	12.77	12.12	11.55	11.13
8	58.43	33.72	24.18	19.28	16.38	14.46	13.12	12.08	11.33	10.73	10.24	9.84
9	55.09	31.54	22.47	17.82	15.07	13.29	12.03	11.04	10.34	9.77	9.27	8.90
10	52.73	29.98	21.26	16.79	14.15	12.40	11.16	10.25	9.57	9.00	8.56	8.20
11	50.85	28.75	20.23	15.92	13.38	11.70	10.53	9.66	8.97	8.45	8.02	7.65
12	49.27	27.73	19.48	15.31	12.83	11.16	10.02	9.17	8.51	8.00	7.56	7.23
13	48.01	26.93	18.80	14.75	12.33	10.71	9.60	8.74	8.09	7.60	7.19	6.85
14	46.91	26.23	18.31	14.29	11.92	10.33	9.24	8.40	7.77	7.29	6.87	6.56
15	46.09	25.69	17.84	13.89	11.52	10.00	8.93	8.14	7.52	7.05	6.63	6.33
16	45.31	25.20	17.45	13.56	11.25	9.74	8.67	7.88	7.25	6.79	6.40	6.08
17	44.67	24.71	17.14	13.30	11.01	9.49	8.46	7.66	7.07	6.61	6.21	5.91
18	44.04	24.39	16.83	13.03	10.78	9.28	8.26	7.47	6.88	6.43	6.04	5.75
19	43.58	24.07	16.58	12.82	10.59	9.13	8.10	7.34	6.73	6.29	5.90	5.61
20	43.12	23.75	16.33	12.61	10.43	8.96	7.96	7.19	6.61	6.15	5.79	5.48
25	41.35	22.64	15.47	11.89	9.76	8.37	7.36	6.65	6.09	5.64	5.28	5.00
30	40.30	21.93	14.93	11.43	9.35	7.97	7.01	6.29	5.75	5.32	4.98	4.70
40	39.05	21.12	14.30	10.87	8.84	7.51	6.58	5.88	5.37	4.95	4.60	4.32
50	38.26	20.62	13.89	10.55	8.55	7.25	6.31	5.63	5.12	4.72	4.38	4.13
60	37.79	20.23	13.65	10.30	8.36	7.07	6.16	5.49	4.97	4.59	4.25	3.96
120	36.69	19.51	13.04	9.82	7.90	6.66	5.76	5.11	4.61	4.24	3.91	3.64

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	35.49	34.84	34.20	33.64	33.16	32.82	31.27	30.33	29.14	28.47	28.01	27.00
4	21.55	21.11	20.66	20.33	20.01	19.68	18.65	17.98	17.19	16.76	16.45	15.68
5	15.82	15.38	15.07	14.76	14.50	14.28	13.46	12.91	12.21	11.84	11.64	11.02
6	12.69	12.35	12.05	11.78	11.55	11.36	10.65	10.18	9.58	9.26	9.08	8.57
7	10.75	10.47	10.20	9.96	9.75	9.58	8.94	8.52	7.98	7.69	7.51	7.02
8	9.49	9.20	8.95	8.76	8.57	8.38	7.76	7.37	6.92	6.66	6.46	6.01
9	8.56	8.29	8.03	7.85	7.67	7.52	6.97	6.55	6.11	5.86	5.69	5.29
10	7.88	7.63	7.37	7.20	7.03	6.86	6.31	5.96	5.56	5.30	5.13	4.75
11	7.37	7.09	6.87	6.68	6.52	6.39	5.83	5.50	5.07	4.82	4.70	4.31
12	6.93	6.69	6.45	6.27	6.11	5.98	5.44	5.10	4.71	4.49	4.33	3.96
13	6.56	6.33	6.12	5.94	5.79	5.66	5.16	4.80	4.42	4.21	4.05	3.68
14	6.28	6.05	5.82	5.67	5.52	5.37	4.87	4.56	4.17	3.95	3.79	3.46
15	6.05	5.83	5.60	5.43	5.29	5.14	4.65	4.35	3.96	3.74	3.59	3.24
16	5.83	5.59	5.39	5.23	5.08	4.96	4.45	4.16	3.78	3.57	3.42	3.07
17	5.64	5.43	5.21	5.05	4.91	4.77	4.29	4.00	3.63	3.42	3.28	2.93
18	5.48	5.27	5.06	4.90	4.76	4.62	4.15	3.84	3.50	3.27	3.13	2.79
19	5.33	5.11	4.91	4.75	4.61	4.48	4.03	3.71	3.35	3.15	3.01	2.67
20	5.22	4.99	4.80	4.62	4.49	4.35	3.92	3.62	3.25	3.03	2.90	2.58
25	4.73	4.51	4.34	4.17	4.04	3.93	3.49	3.19	2.84	2.65	2.52	2.19
30	4.44	4.24	4.05	3.88	3.76	3.66	3.21	2.92	2.58	2.40	2.27	1.94
40	4.07	3.89	3.70	3.55	3.43	3.32	2.88	2.60	2.28	2.08	1.94	1.61
50	3.88	3.68	3.52	3.36	3.23	3.11	2.70	2.43	2.08	1.89	1.77	1.43
60	3.72	3.54	3.37	3.21	3.10	2.98	2.56	2.29	1.95	1.77	1.64	1.31
120	3.40	3.21	3.06	2.91	2.77	2.67	2.25	1.99	1.65	1.46	1.32	0.99

TABLE II (continued)

$1 - \alpha = 0.95,\ \beta = 0.99$

n_2^*	n_1^* = 3	4	5	6	7	8	9	10	11	12	13	14
3	452.95	284.62	217.24	182.31	161.08	147.11	137.00	129.63	123.73	119.03	115.22	111.88
4	250.35	151.74	113.03	93.11	81.04	73.09	67.55	63.35	60.20	57.56	55.56	53.75
5	177.90	105.07	76.90	62.38	53.75	47.98	43.95	41.09	38.72	36.96	35.51	34.25
6	142.68	82.71	59.61	47.84	40.84	36.22	33.03	30.59	28.79	27.36	26.14	25.17
7	122.10	69.78	49.62	39.58	33.53	29.59	26.79	24.76	23.20	21.97	20.92	20.08
8	108.96	61.56	43.44	34.30	28.96	25.35	22.88	21.05	19.65	18.55	17.67	16.95
9	99.77	55.78	39.13	30.67	25.75	22.52	20.23	18.55	17.27	16.26	15.40	14.73
10	93.09	51.61	36.01	28.11	23.48	20.44	18.29	16.74	15.57	14.57	13.81	13.19
11	88.10	48.51	33.64	26.12	21.69	18.80	16.81	15.33	14.19	13.33	12.61	11.98
12	83.99	46.02	31.69	24.59	20.43	17.62	15.71	14.29	13.20	12.38	11.65	11.09
13	80.82	44.04	30.25	23.34	19.28	16.62	14.78	13.42	12.37	11.57	10.95	10.34
14	78.09	42.42	29.02	22.36	18.42	15.84	14.06	12.74	11.72	10.95	10.30	9.78
15	75.93	41.06	28.05	21.50	17.66	15.15	13.42	12.13	11.18	10.43	9.77	9.27
16	73.98	39.93	27.19	20.79	17.04	14.62	12.90	11.68	10.68	9.95	9.38	8.85
17	72.32	38.98	26.47	20.19	16.52	14.11	12.45	11.23	10.32	9.60	8.98	8.50
18	70.94	38.12	25.82	19.65	16.08	13.72	12.09	10.89	9.99	9.26	8.68	8.21
19	69.76	37.35	25.23	19.21	15.65	13.37	11.73	10.58	9.67	8.98	8.38	7.92
20	68.72	36.66	24.71	18.77	15.35	13.02	11.45	10.31	9.45	8.71	8.18	7.69
25	64.72	34.28	22.97	17.33	14.01	11.87	10.36	9.29	8.45	7.81	7.28	6.85
30	62.24	32.77	21.83	16.38	13.19	11.14	9.70	8.68	7.87	7.23	6.75	6.31
40	59.27	30.99	20.48	15.28	12.26	10.30	8.93	7.96	7.18	6.58	6.10	5.68
50	57.60	29.96	19.73	14.65	11.74	9.80	8.48	7.53	6.78	6.19	5.73	5.34
60	56.54	29.30	19.24	14.29	11.36	9.49	8.22	7.26	6.53	5.97	5.49	5.12
120	53.74	27.64	18.05	13.28	10.55	8.73	7.53	6.61	5.91	5.36	4.93	4.56

n_2^*	15	16	17	18	19	20	25	30	40	50	60	120
3	109.62	107.58	105.54	103.78	102.32	101.14	96.20	93.25	89.50	87.45	85.96	82.81
4	52.28	51.14	50.01	49.19	48.38	47.57	44.96	43.31	41.37	40.26	39.45	37.56
5	33.29	32.37	31.65	30.97	30.39	29.91	28.07	26.92	25.44	24.64	24.16	22.85
6	24.35	23.68	23.10	22.56	22.09	21.70	20.22	19.32	18.14	17.51	17.14	16.17
7	19.38	18.81	18.34	17.87	17.47	17.14	15.87	15.07	14.09	13.55	13.24	12.35
8	16.32	15.75	15.30	14.94	14.59	14.26	13.13	12.44	11.60	11.12	10.79	10.06
9	14.16	13.69	13.26	12.93	12.60	12.32	11.31	10.62	9.87	9.43	9.14	8.47
10	12.66	12.22	11.82	11.51	11.20	10.93	9.96	9.38	8.69	8.28	7.99	7.37
11	11.52	11.06	10.72	10.39	10.10	9.88	8.97	8.43	7.73	7.33	7.13	6.49
12	10.61	10.21	9.85	9.57	9.30	9.09	8.19	7.67	7.03	6.66	6.44	5.84
13	9.90	9.53	9.25	8.91	8.64	8.44	7.61	7.08	6.46	6.12	5.89	5.33
14	9.34	8.97	8.63	8.38	8.12	7.89	7.09	6.59	5.98	5.65	5.43	4.92
15	8.84	8.46	8.16	7.91	7.68	7.42	6.67	6.18	5.59	5.27	5.07	4.54
16	8.46	8.08	7.80	7.52	7.28	7.07	6.31	5.86	5.27	4.96	4.75	4.25
17	8.08	7.75	7.44	7.20	6.96	6.76	6.02	5.58	5.02	4.70	4.50	4.01
18	7.80	7.47	7.17	6.93	6.70	6.48	5.78	5.30	4.78	4.46	4.26	3.78
19	7.52	7.22	6.90	6.67	6.47	6.24	5.54	5.09	4.58	4.25	4.06	3.59
20	7.33	6.98	6.71	6.46	6.24	6.04	5.35	4.92	4.39	4.06	3.89	3.42
25	6.48	6.15	5.91	5.67	5.46	5.31	4.63	4.22	3.72	3.45	3.26	2.82
30	5.96	5.67	5.41	5.18	4.98	4.83	4.19	3.78	3.30	3.05	2.88	2.45
40	5.35	5.07	4.83	4.61	4.43	4.28	3.67	3.29	2.84	2.57	2.39	1.99
50	5.02	4.73	4.52	4.31	4.13	3.95	3.37	3.01	2.55	2.29	2.14	1.72
60	4.78	4.52	4.29	4.08	3.91	3.75	3.18	2.82	2.36	2.12	1.97	1.55
120	4.26	3.99	3.78	3.58	3.40	3.27	2.70	2.36	1.92	1.68	1.53	1.11

TABLE II (continued)

$1 - \alpha = 0.90,\ \beta = 0.90$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	49.44	34.61	28.14	24.54	22.33	20.77	19.63	18.81	18.14	17.62	17.17	16.77
4	37.58	25.62	20.37	17.51	15.74	14.51	13.65	12.98	12.48	12.05	11.71	11.41
5	31.95	21.36	16.74	14.24	12.69	11.63	10.86	10.29	9.80	9.47	9.15	8.92
6	28.72	18.89	14.69	12.34	10.93	9.95	9.25	8.74	8.30	7.97	7.71	7.48
7	26.56	17.31	13.33	11.18	9.80	8.89	8.21	7.73	7.32	7.05	6.78	6.56
8	25.10	16.24	12.42	10.33	9.02	8.16	7.51	7.04	6.67	6.36	6.11	5.90
9	24.03	15.40	11.73	9.70	8.47	7.61	6.99	6.54	6.19	5.89	5.64	5.45
10	23.23	14.81	11.19	9.22	7.98	7.17	6.58	6.11	5.80	5.51	5.25	5.06
11	22.56	14.31	10.75	8.86	7.65	6.83	6.25	5.83	5.50	5.22	4.97	4.79
12	22.06	13.90	10.42	8.54	7.36	6.56	6.00	5.58	5.26	4.98	4.74	4.54
13	21.57	13.55	10.17	8.29	7.11	6.36	5.77	5.36	5.05	4.78	4.54	4.34
14	21.19	13.26	9.94	8.08	6.92	6.15	5.60	5.20	4.86	4.60	4.36	4.20
15	20.91	13.03	9.70	7.92	6.76	6.00	5.46	5.04	4.74	4.48	4.24	4.06
16	20.62	12.83	9.52	7.75	6.63	5.88	5.33	4.91	4.61	4.35	4.12	3.94
17	20.42	12.69	9.40	7.59	6.49	5.74	5.20	4.78	4.48	4.23	4.01	3.82
18	20.22	12.55	9.28	7.48	6.39	5.65	5.09	4.70	4.38	4.12	3.91	3.75
19	20.02	12.41	9.17	7.38	6.30	5.57	5.01	4.63	4.31	4.04	3.84	3.66
20	19.84	12.27	9.05	7.28	6.21	5.48	4.93	4.55	4.23	3.97	3.77	3.59
25	19.27	11.79	8.66	6.95	5.90	5.18	4.65	4.27	3.95	3.71	3.51	3.32
30	18.87	11.49	8.41	6.73	5.70	4.96	4.45	4.07	3.76	3.51	3.32	3.15
40	18.39	11.13	8.10	6.45	5.41	4.74	4.20	3.85	3.53	3.28	3.10	2.93
50	18.09	10.94	7.93	6.30	5.27	4.61	4.08	3.71	3.39	3.17	2.99	2.81
60	17.89	10.80	7.82	6.16	5.17	4.48	4.00	3.62	3.32	3.09	2.90	2.73
120	17.42	10.45	7.53	5.90	4.92	4.24	3.76	3.39	3.10	2.87	2.68	2.52

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	16.51	16.22	16.01	15.77	15.62	15.46	14.89	14.50	14.05	13.77	13.61	13.23
4	11.18	10.96	10.74	10.62	10.48	10.36	9.91	9.61	9.25	9.06	8.88	8.59
5	8.69	8.50	8.35	8.22	8.09	8.00	7.59	7.32	7.03	6.85	6.73	6.44
6	7.28	7.11	6.96	6.85	6.74	6.65	6.29	6.04	5.77	5.60	5.47	5.23
7	6.35	6.22	6.09	5.96	5.85	5.77	5.46	5.23	4.95	4.78	4.69	4.44
8	5.73	5.58	5.45	5.33	5.24	5.16	4.84	4.63	4.37	4.23	4.14	3.90
9	5.28	5.14	5.02	4.90	4.81	4.74	4.41	4.19	3.94	3.80	3.72	3.48
10	4.92	4.76	4.65	4.53	4.44	4.38	4.08	3.87	3.61	3.48	3.39	3.17
11	4.65	4.49	4.38	4.27	4.18	4.10	3.81	3.60	3.37	3.22	3.14	2.92
12	4.41	4.25	4.14	4.04	3.95	3.89	3.57	3.38	3.16	3.02	2.94	2.73
13	4.21	4.06	3.96	3.85	3.77	3.71	3.40	3.21	2.98	2.84	2.76	2.57
14	4.07	3.92	3.81	3.71	3.63	3.55	3.25	3.07	2.85	2.72	2.62	2.42
15	3.93	3.79	3.68	3.58	3.48	3.40	3.10	2.93	2.71	2.58	2.49	2.28
16	3.81	3.67	3.57	3.47	3.37	3.29	3.00	2.83	2.60	2.47	2.38	2.18
17	3.70	3.56	3.46	3.36	3.26	3.18	2.89	2.72	2.50	2.37	2.28	2.09
18	3.61	3.49	3.37	3.27	3.17	3.10	2.79	2.63	2.41	2.28	2.19	2.00
19	3.52	3.40	3.28	3.18	3.09	3.01	2.73	2.56	2.34	2.21	2.13	1.92
20	3.45	3.33	3.22	3.12	3.02	2.95	2.67	2.51	2.29	2.15	2.08	1.86
25	3.18	3.07	2.96	2.86	2.79	2.72	2.44	2.26	2.03	1.91	1.81	1.61
30	3.01	2.90	2.79	2.70	2.61	2.54	2.26	2.09	1.85	1.74	1.66	1.45
40	2.80	2.70	2.59	2.50	2.40	2.33	2.05	1.87	1.66	1.52	1.44	1.23
50	2.68	2.56	2.45	2.37	2.28	2.20	1.93	1.75	1.54	1.40	1.33	1.09
60	2.58	2.46	2.36	2.27	2.19	2.12	1.84	1.68	1.46	1.32	1.23	1.01
120	2.37	2.25	2.15	2.07	1.98	1.92	1.65	1.46	1.24	1.09	1.00	0.77

TABLE II (continued)

$1 - \alpha = 0.90,\ \beta = 0.95$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	85.12	59.10	47.74	41.55	37.63	34.98	33.01	31.57	30.45	29.50	28.74	28.09
4	59.10	39.83	31.49	26.90	24.09	22.19	20.78	19.74	18.92	18.27	17.73	17.26
5	47.74	31.49	24.50	20.73	18.37	16.79	15.63	14.78	14.04	13.55	13.07	12.73
6	41.55	26.90	20.73	17.32	15.29	13.86	12.82	12.07	11.45	11.00	10.62	10.29
7	37.63	24.09	18.37	15.29	13.36	12.05	11.11	10.40	9.84	9.42	9.05	8.75
8	34.98	22.19	16.79	13.86	12.05	10.83	9.95	9.25	8.76	8.35	8.00	7.72
9	33.01	20.78	15.63	12.82	11.11	9.95	9.11	8.45	7.99	7.60	7.27	7.00
10	31.57	19.74	14.78	12.07	10.40	9.25	8.45	7.82	7.41	7.03	6.69	6.44
11	30.45	18.92	14.04	11.45	9.84	8.76	7.99	7.41	6.95	6.59	6.26	6.01
12	29.50	18.27	13.55	11.00	9.42	8.35	7.60	7.03	6.59	6.24	5.92	5.68
13	28.74	17.73	13.07	10.62	9.05	8.00	7.27	6.69	6.26	5.92	5.60	5.37
14	28.09	17.26	12.73	10.29	8.75	7.72	7.00	6.44	6.01	5.68	5.37	5.15
15	27.62	16.90	12.40	9.99	8.45	7.47	6.77	6.24	5.83	5.50	5.20	4.98
16	27.16	16.58	12.11	9.74	8.28	7.29	6.57	6.02	5.61	5.29	5.00	4.78
17	26.78	16.26	11.90	9.53	8.09	7.11	6.40	5.86	5.46	5.14	4.85	4.64
18	26.40	16.05	11.69	9.35	7.93	6.96	6.26	5.72	5.33	4.99	4.73	4.52
19	26.11	15.85	11.52	9.20	7.80	6.84	6.14	5.62	5.22	4.89	4.63	4.42
20	25.85	15.65	11.36	9.06	7.67	6.72	6.03	5.51	5.12	4.79	4.53	4.33
25	24.83	14.93	10.78	8.56	7.22	6.28	5.61	5.12	4.72	4.40	4.15	3.94
30	24.17	14.44	10.38	8.22	6.91	5.99	5.32	4.83	4.45	4.14	3.90	3.71
40	23.42	13.90	9.93	7.82	6.51	5.65	4.98	4.53	4.16	3.84	3.61	3.43
50	22.94	13.59	9.66	7.59	6.30	5.45	4.80	4.33	3.95	3.68	3.45	3.26
60	22.65	13.34	9.50	7.41	6.16	5.32	4.69	4.21	3.86	3.57	3.35	3.13
120	22.00	12.87	9.08	7.07	5.81	4.98	4.39	3.93	3.58	3.31	3.07	2.88

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	27.62	27.16	26.78	26.40	26.11	25.85	24.83	24.17	23.42	22.94	22.65	22.00
4	16.90	16.58	16.26	16.05	15.85	15.65	14.93	14.44	13.90	13.59	13.34	12.87
5	12.40	12.11	11.90	11.69	11.52	11.36	10.78	10.38	9.93	9.66	9.50	9.08
6	9.99	9.74	9.53	9.35	9.20	9.06	8.56	8.22	7.82	7.59	7.41	7.07
7	8.45	8.28	8.09	7.93	7.80	7.67	7.22	6.91	6.51	6.30	6.16	5.81
8	7.47	7.29	7.11	6.96	6.84	6.72	6.28	5.99	5.65	5.45	5.32	4.98
9	6.77	6.57	6.40	6.26	6.14	6.03	5.61	5.32	4.98	4.80	4.69	4.39
10	6.24	6.02	5.86	5.72	5.62	5.51	5.12	4.83	4.53	4.33	4.21	3.93
11	5.83	5.61	5.46	5.33	5.22	5.12	4.72	4.45	4.16	3.95	3.86	3.58
12	5.50	5.29	5.14	4.99	4.89	4.79	4.40	4.14	3.84	3.68	3.57	3.31
13	5.20	5.00	4.85	4.73	4.63	4.53	4.15	3.90	3.61	3.45	3.35	3.07
14	4.98	4.78	4.64	4.52	4.42	4.33	3.94	3.71	3.43	3.26	3.13	2.88
15	4.81	4.62	4.47	4.33	4.22	4.13	3.74	3.52	3.24	3.08	2.95	2.69
16	4.62	4.43	4.29	4.17	4.06	3.97	3.59	3.38	3.10	2.94	2.82	2.56
17	4.47	4.29	4.15	4.02	3.91	3.82	3.45	3.24	2.97	2.81	2.69	2.44
18	4.33	4.17	4.02	3.88	3.77	3.69	3.32	3.11	2.85	2.69	2.58	2.33
19	4.22	4.06	3.91	3.77	3.67	3.58	3.24	3.02	2.74	2.58	2.48	2.22
20	4.13	3.97	3.82	3.69	3.58	3.49	3.16	2.94	2.66	2.49	2.41	2.15
25	3.74	3.59	3.45	3.32	3.24	3.16	2.84	2.61	2.33	2.18	2.08	1.83
30	3.52	3.38	3.24	3.11	3.02	2.94	2.61	2.39	2.11	1.97	1.87	1.62
40	3.24	3.10	2.97	2.85	2.74	2.66	2.33	2.11	1.86	1.71	1.61	1.35
50	3.08	2.94	2.81	2.69	2.58	2.49	2.18	1.97	1.71	1.56	1.46	1.19
60	2.95	2.82	2.69	2.58	2.48	2.41	2.08	1.87	1.61	1.46	1.34	1.09
120	2.69	2.56	2.44	2.33	2.22	2.15	1.83	1.62	1.35	1.19	1.09	0.82

TABLE II (continued)

$1 - \alpha = 0.90,\ \beta = 0.99$

n_2^* \ n_1^*	3	4	5	6	7	8	9	10	11	12	13	14
3	270.21	186.05	149.64	129.73	117:25	108.77	102.43	97.92	94.30	91.31	88.71	86.53
4	149.68	99.61	78.07	66.43	59.26	54.38	50.77	48.16	46.12	44.41	43.05	41.86
5	106.43	69.11	53.23	44.69	39.35	35.77	33.20	31.35	29.75	28.64	27.55	26.81
6	85.38	54.34	41.32	34.27	30.00	27.05	24.95	23.40	22.14	21.23	20.44	19.74
7	73.09	45.90	34.48	28.41	24.68	22.11	20.25	18.96	17.87	17.08	16.39	15.79
8	65.23	40.51	30.16	24.66	21.32	18.99	17.35	16.12	15.20	14.43	13.81	13.29
9	59.77	36.75	27.23	22.06	18.97	16.85	15.33	14.20	13.35	12.64	12.07	11.59
10	55.74	33.98	25.04	20.21	17.25	15.25	13.85	12.77	12.04	11.38	10.80	10.35
11	52.75	31.94	23.34	18.79	15.95	14.08	12.76	11.76	11.00	10.39	9.84	9.42
12	50.30	30.31	22.05	17.67	15.01	13.18	11.91	10.96	10.22	9.64	9.12	8.71
13	48.37	28.99	21.04	16.81	14.15	12.42	11.20	10.27	9.57	9.01	8.53	8.12
14	46.76	27.91	20.18	16.10	13.52	11.84	10.65	9.75	9.07	8.53	8.06	7.68
15	45.51	27.01	19.49	15.47	12.96	11.32	10.17	9.30	8.66	8.14	7.66	7.29
16	44.34	26.27	18.87	14.93	12.55	10.95	9.78	8.92	8.27	7.76	7.32	6.96
17	43.34	25.65	18.38	14.47	12.14	10.57	9.42	8.59	7.97	7.47	7.02	6.67
18	42.52	25.09	17.93	14.09	11.83	10.28	9.16	8.34	7.73	7.19	6.80	6.45
19	41.79	24.59	17.54	13.79	11.53	10.01	8.90	8.09	7.50	6.99	6.58	6.24
20	41.19	24.15	17.18	13.49	11.29	9.76	8.67	7.90	7.32	6.79	6.41	6.08
25	38.87	22.60	16.00	12.48	10.36	8.91	7.90	7.15	6.54	6.09	5.73	5.39
30	37.33	21.58	15.19	11.79	9.74	8.37	7.37	6.67	6.09	5.63	5.29	4.99
40	35.55	20.40	14.22	11.00	9.03	7.75	6.76	6.12	5.57	5.11	4.78	4.50
50	34.53	19.75	13.72	10.54	8.65	7.37	6.44	5.79	5.23	4.83	4.52	4.22
60	33.90	19.32	13.39	10.28	8.38	7.15	6.26	5.57	5.06	4.65	4.32	4.04
120	32.22	18.22	12.56	9.56	7.76	6.53	5.73	5.08	4.60	4.19	3.87	3.61

n_2^* \ n_1^*	15	16	17	18	19	20	25	30	40	50	60	120
3	85.31	83.84	82.64	81.44	80.55	79.66	76.40	74.31	71.93	70.45	69.52	67.47
4	41.00	40.17	39.35	38.84	38.33	37.82	35.98	34.79	33.46	32.64	32.00	30.83
5	26.08	25.48	24.99	24.52	24.15	23.79	22.48	21.65	20.68	20.09	19.73	18.81
6	19.17	18.69	18.28	17.90	17.60	17.30	16.25	15.60	14.81	14.35	14.00	13.33
7	15.23	14.88	14.55	14.23	13.97	13.71	12.81	12.22	11.51	11.11	10.87	10.24
8	12.85	12.48	12.16	11.87	11.64	11.43	10.62	10.12	9.46	9.11	8.89	8.33
9	11.19	10.85	10.57	10.31	10.09	9.88	9.11	8.63	8.05	7.72	7.53	7.02
10	10.02	9.65	9.40	9.15	8.95	8.77	8.08	7.61	7.07	6.76	6.56	6.09
11	9.10	8.76	8.51	8.28	8.09	7.92	7.26	6.82	6.35	6.00	5.85	5.39
12	8.41	8.08	7.85	7.62	7.44	7.28	6.62	6.23	5.73	5.46	5.31	4.87
13	7.85	7.53	7.31	7.09	6.92	6.77	6.13	5.76	5.28	5.02	4.86	4.45
14	7.41	7.10	6.88	6.68	6.51	6.36	5.74	5.37	4.92	4.67	4.48	4.10
15	7.02	6.70	6.51	6.31	6.13	5.95	5.37	5.01	4.58	4.33	4.17	3.78
16	6.70	6.41	6.20	6.01	5.82	5.66	5.09	4.76	4.33	4.09	3.91	3.55
17	6.41	6.13	5.93	5.73	5.54	5.41	4.84	4.51	4.11	3.86	3.69	3.33
18	6.17	5.92	5.69	5.50	5.32	5.17	4.63	4.30	3.90	3.67	3.51	3.16
19	5.96	5.73	5.49	5.30	5.14	4.99	4.45	4.14	3.74	3.48	3.35	2.98
20	5.80	5.55	5.34	5.15	4.97	4.85	4.31	3.99	3.60	3.34	3.23	2.86
25	5.13	4.90	4.70	4.52	4.39	4.27	3.77	3.45	3.05	2.85	2.69	2.36
30	4.73	4.52	4.32	4.15	4.00	3.88	3.40	3.09	2.71	2.51	2.38	2.05
40	4.26	4.05	3.87	3.71	3.54	3.43	2.97	2.67	2.32	2.11	1.98	1.67
50	3.98	3.78	3.60	3.45	3.30	3.16	2.73	2.45	2.09	1.89	1.77	1.44
60	3.79	3.60	3.42	3.27	3.13	3.03	2.58	2.30	1.95	1.76	1.61	1.29
120	3.37	3.19	3.01	2.87	2.73	2.63	2.20	1.92	1.58	1.37	1.26	0.93

n_2^*	120	55	38	30	26	23	21	19	18	17	15	13	12	11	10
n_1^*	11	12	13	14	15	16	17	18	19	20	24	30	36	57	112

Thus $n_1 + n_2$ is minimized for $n_1 = 19$, $n_2 = 20$ or $n_1 = 18$, $n_2 = 21$.

Note that if the confidence interval were to be placed on $(\sigma_1^2 + \sigma_2^2)/\sigma_1^2$ and if d were equal to 2, the same sample sizes would meet the requirements.

Example 2. Using the same values as in Example 1, if the cost of n_1 is \$2.50 per sample and the cost of n_2 is \$1.00 we may obtain the following results as to the samples required to minimize the total cost:

$(n_2^*;n_1^*)$	55;12	39;13	30;14	26;15	24;16	21;17	19;18
C	149.50	110.50	89.00	80.00	76.00	69.50	65.50
$(n_2^*;n_1^*)$	18;19	17;20	15;24	13;30	12;36	11;57	10;112
C	64.00	62.50	61.50	162.50	66.00	94.00	137.00

Thus C is minimized when $n_1 = 16$ and $n_2 = 25$.

Example 3. If we want the width of the interval to be less than p per cent of σ_2^2/σ_1^2 with width coefficient β, then a solution may be found in one step. The derivation is as follows:

We desire

$$P\left[w < \frac{p}{100}\frac{\sigma_2^2}{\sigma_1^2}\right] = \beta. \qquad (4.6)$$

Specifically, we desire values of n_1 and n_2 such that

$$P\left\{\frac{s_2^2}{s_1^2}\left[F_{\alpha/2}(n_1,n_2) - \frac{1}{F_{\alpha/2}(n_2,n_1)}\right] \leq \frac{p}{100}\frac{\sigma_2^2}{\sigma_1^2}\right\} \geq \beta,$$

or

$$P\left\{\frac{s_2^2}{\sigma_2^2}\frac{\sigma_1^2}{s_1^2} < \frac{p}{100}\left[F_{\alpha/2}(n_1,n_2) - \frac{1}{F_{\alpha/2}(n_2,n_1)}\right]^{-1}\right\} \geq \beta.$$

It is known that

$$P\left[\frac{s_2^2}{\sigma_2^2}\frac{\sigma_1^2}{s_1^2} < F_{1-\beta}(n_2,n_1)\right] = \beta.$$

Therefore (4.6) is satisfied when we choose the smallest integers n_1 and n_2 so that

$$\frac{p}{100}[C(n_1,n_2]^{-1} \geq F_{1-\beta}(n_2,n_1),$$

or

$$\frac{p}{100} \geq F_{1-\beta}(n_2,n_1)C(n_1,n_2),$$

or

$$\frac{p}{100} \geq Q(n_1,n_2).$$

Thus, Table II provides the solution.

Suppose it is desired to set a 95 per cent confidence interval on σ_2^2/σ_1^2 with width less than 584 per cent of σ_2^2/σ_1^2 with β = 99 per cent. In this case, we have $1 - \alpha = 0.95$, $\beta = 0.99$, and $p/100 = 5.84$, and the choices of values found in the tables are as follows:

n_2^*	120	50	31	28	24	23	22	21	18	17	15	14	12
n_1^*	12	14	15	17	18	19	20	21	24	27	33	39	80

There are several choices of n_1^* and n_2^* such that the minimum of $n_1^* + n_2^*$ is 42.

CHAPTER V

THE EXPECTED SIZE OF A SAMPLE FOR THE DESIRED WIDTH INTERVAL ON THE RATIO OF VARIANCES

We shall derive the expected size of a sample necessary for a β^2 width coefficient and a $1 - \alpha$ confidence interval on the ratio of two variances from normal populations.

We shall slightly alter the procedure of selecting n_1 and n_2 in order to compute the expected value of $n_1 + n_2$ in establishing a β^2 width coefficient and a $1 - \alpha$ confidence interval on σ_2^2/σ_1^2. If the procedure does not permit n_1 and n_2 to be equal, then we shall take the minimum $n_1 + n_2$ and let both n_1 and n_2 be equal to the larger.

The technique used is to choose the smallest integral values of n_1 and n_2 so that

$$F_{1-\beta}(n_2,n_1)C(n_1,n_2) \leq \frac{dF_\beta(m_2,m_1)}{z},$$

where

$$C(n_1,n_2) = F_{\alpha/2}(n_1,n_2) - \frac{1}{F_{\alpha/2}(n_1,n_2)}.$$

Since we have chosen n_1 to be equal to n_2, we shall use the notation $G(n_1,n_2) = G(n)$. Solving for z, we obtain

$$z \leq f_1(n) = \frac{dF_\beta(m_2,m_1)}{F_{1-\beta}(n)C(n)},$$

or

$$z \leq f_1(k) = \frac{dF_\beta(m_2,m_1)}{F_{1-\beta}(k)C(k)},$$

where

$$k - 1 < n \leq k.$$

Note that z is monotonically increasing in n, and k is the smallest integer $\geq$ n.

From Chapter III we obtained

$$E(k) = \sum_{u=1}^{\infty} uP(u - 1 < n < u)$$

$$= \lim_{N \to \infty} [-NP(n \geq N) + 1 + \sum_{k=1}^{N-1} P(n \geq k)]. \tag{5.1}$$

Because z is a monotonically increasing function in n, then

$$P(n \geq N) = P[f_1(n) \geq f_1(N)].$$

Since $(\sigma_1^2/\sigma_2^2)z$ follows the central F distribution with q_2 and q_1 degrees of freedom, we have

$$\lim_{N \to \infty} NP(n \geq N) = \lim_{N \to \infty} N \frac{[(q_1 + q_2 - 2)/2]!}{[(q_1 - 2)/2]![(q_2 - 2)/2]!} \left(\frac{q_2}{q_1}\right)^{q_2/2}$$

$$\times \int_{\theta f_1(N)}^{\infty} \frac{x^{(q_2-2)/2}}{(1 + q_2x/q_1)^{[(q_1+q_2)/2]}} \, dx, \tag{5.2}$$

where $q_1 = m_1 - 1$, $q_2 = m_2 - 2$, and

$$f_1(N) = \frac{F_\beta(m_2, m_1)}{C(N)F_{1-\beta}(N)} .$$

In order to evaluate the above integral we shall show that

$$\frac{x^{(q_2-2)/2}}{(1 + q_2x/q_1)^{[(q_1+q_2)/2]}} \leq \frac{x}{(1 + x)^4} \tag{5.3}$$

for all $x > x = x^*$.

Let us assume the assertion to be true for some values of x. We shall find the values of x for which the inequality holds. Suppose $q_2 \geq q_1$, or $q_2 = cq_1$, then

$$\frac{x^{(cq_1-2)/2}}{(1 + cx)^{q_1[(c+1)/2]}} \leq \frac{x}{(1 + x)^4}.$$

Since

$$\frac{x}{(1 + x)^4} \geq \frac{x}{(1 + cx)^4},$$

we may write

$$\frac{x^{(cq_1-4)/2}}{(1 + cx)^{[(cq_1-4)/2]}(1 + cx)^{[(q_1-4)/2]}} \leq 1$$

which is true for all $x \geq 0$.

When $q_1 > q_2$ or $q_1 = cq_2$, then we desire the values of x such that

$$\frac{x^{(q_2-2)/2}}{\{1 + (x/c)]^{[q_2(c+1)/2]}} \leq \frac{x}{(1 + x)^4}.$$

Since

$$\frac{x}{(1 + x)^4} \geq \frac{x}{(c + x)^4},$$

we may write

$$\frac{x^{(q_2-4)/2}\,r}{(c + x)^{(q_2-4)/2}(c + x)^{(cq_2-4)/2}} \leq 1,$$

where

$$r = c^{q_2(c+1)/2}.$$

Since $x/(c + x) < 1$ we obtain

$$r \leq (c + x)^{(cq_2-4)/2},$$

or

$$r^{2/cq_2-4)} - c \leq x,$$

where $q_2 > 4$.

Therefore, there exists a value of $x = x^*$ so that (5.3) holds for all $x > x^*$.

If $f_2(x) < g_2(x)$ for all $x > x^*$,

then

$$\lim_{x \to \infty} f_2(x) \leq \lim_{x \to \infty} g_2(x),$$

provided the limits exist.

We shall use the following notation:

$$c(q_1,q_2) = \frac{[(q_1+q_2-2)/2]!}{[(q_1-2)/2]![(q_2-2)/2]!}\left(\frac{q_2}{q_1}\right)^{q_2/2},$$

and

$$d(q_2,q_1) = \frac{\sigma_2^2}{\sigma_1^2}\, dF_{1-\beta}(m_2,m_1).$$

We may write

$$\int_{x=x^*}^{\infty} = \frac{d(q_2,q_1)}{C(N)F_{1-\beta}(N)} \quad \frac{x^{(q_2-2)/2}}{[1 + (q_2x/q_1)]^{(q_1+q_2)/2}}\, dx$$

$$\leq \int_{x=x^*}^{\infty} = \frac{d(q_2,q_1)}{C(N)F_{1-\beta}(N)} \quad \frac{x}{(1 + x)^4}\, dx, \tag{5.4}$$

where

$$x^* = 0 \text{ if } q_2 > q_1,$$

or

$$x^* \geq c^{q_2(c+1)/(cq_2-4)} - c \text{ if } q_1 > q_2.$$

Because

$$\frac{d(q_2, q_1)}{C(N)F_{1-\beta}(N)} \to \infty$$

as $N \to \infty$, there exists a value of x^* such that

$$\frac{d(q_2, q_1)}{F_{1-\beta}(N)C(N)} \geq x^*$$

for all N greater than N^*. Therefore.

$$\lim_{N \to \infty} c(q_1, q_2) N \int_{\frac{d(q_2,q_1)}{C(N)F_{1-\beta}(N)}}^{\infty} \frac{x^{(q_2-2)/2}}{[1 + (q_2 x/q_1)]^{(q_1+q_2)/2}} dx$$

$$\leq \lim_{N \to \infty} c(q_1, q_2) N \int_{\frac{d(q_2,q_1)}{C(N)F_{1-\beta}(N)}}^{\infty} \frac{x}{(1 + x)^4} dx, \qquad (5.5)$$

provided the limits exist.

Cochran (8) has shown that $F_\gamma(N)$ can be approximated by

$10^{c_i/\sqrt{N-c_j}}$,

where c_i and c_j are constants such that $0.40 < c_i < 3$ for $0.70 < \gamma < 0.9999$, and $0.50 < c_j < 3$ for $0.70 < \gamma < 0.9999$. For large N, c_j becomes negligible; hence, we may omit c_j as an approximation term. c_i and c_j are values dependent upon the normal distribution. In the limiting case, the difference between the approximation and the true value is of magnitude $o(1/N^{3/2})$.

Thus if we make use of the fact that

$$F_\gamma(N) = 10^{c_i/\sqrt{N-c_j}} \pm o\left(\frac{1}{N^{3/2}}\right)$$

for large N, we may write (5.5) as

$$\lim_{N\to\infty} c(q_1,q_2)N \int_{g(n)}^{\infty} x(1+x)^{-4}\,dx$$

$$= \lim_{N\to\infty} c(q_1,q_2)\left[\frac{-x}{3(1+x)^3} - \frac{1}{6(1+x)^2}\right]_{x=g(n)}^{x=\infty},$$

where

$$g(n) = \frac{d(q_2,q_1)}{\left[10^{c_1/\sqrt{N}} \pm o\left(\frac{1}{N^{3/2}}\right)\right]\left[10^{c_2/\sqrt{N}} - 10^{-c_2/\sqrt{N}} \pm o\left(\frac{1}{N^{3/2}}\right)\right]}.$$

This may readily be shown to be equal to

$$\lim_{N\to\infty} c(q_1,q_2)\left[\frac{-x}{3(1+x)^3} - \frac{1}{6(1+x)^2}\right]_{x=h(n)}^{x=\infty}$$

$$= \frac{c(q_1,q_2)}{3}\left\{\lim_{N\to\infty}\left[\frac{\dfrac{Nd(q_2,q_1)}{\left[10^{(c_1+c_2)/\sqrt{N}} - 10^{(c_1-c_2)/\sqrt{N}}\right]}}{\left\{1+\dfrac{d(q_2,q_1)}{\left[10^{(c_1+c_2)/\sqrt{N}} - 10^{(c_1-c_2)/\sqrt{N}}\right]}\right\}^3}\right]\right.$$

$$\left. + \lim_{N\to\infty}\left[\frac{N}{2\left\{1 + d(q_2,q_1)/\left[10^{(c_1+c_2)/\sqrt{N}} - 10^{(c_1-c_2)/\sqrt{N}}\right]^2\right\}}\right]\right\}, \quad (5.6)$$

where

$$h(n) = \frac{d(q_2,q_1)}{\left[10^{c_1/\sqrt{N}}\right]\left[10^{c_2/\sqrt{N}} - 10^{-c_2/\sqrt{N}}\right]}.$$

Since

$$\frac{1}{(1 + x)^2} > \frac{x}{(1 + x)^3} \quad \text{for all } x \geq 0,$$

then

$$\lim_{N \to \infty} \frac{N}{(1 + x)^2} \geq \lim_{N\to\infty} \frac{Nx}{(1 + x)^3}, \tag{5.7}$$

or

$$\lim_{N\to\infty} N \int_R \frac{dx}{(1 + x)} \geq \lim_{N\to\infty} N \int_R \frac{x}{(1 + x)^3}\, dx$$

(provided the limit exists). Thus, if we examine the second term in (5.6) and show that its limit exists, then we have also shown that the first term in (5.6) has a limit. We may write the second term as

$$\frac{1}{2} \lim_{N\to\infty} \frac{N\left\{10^{[(c_1+c_2)/\sqrt{N}]} - 10^{[(c_1-c_2)/\sqrt{N}]}\right\}^2}{\left\{10^{[(c_1+c_2)/\sqrt{N}]} - 10^{[(c_1-c_2/\sqrt{N}]} + d(q_2,q_1)\right\}^2} .$$

The limit of the denominator is $2[d(q_2,q_1)]^2$.

The limit of the numerator is found as follows:

$$\lim_{N\to\infty} \frac{\left[10^{2(c_1+c_2)/\sqrt{N}} - (2)\ 10^{2c_1/\sqrt{N}} + 10^{2(c_1-c_2)/\sqrt{N}}\right]}{1/N} = \frac{0}{0}.$$

Since this expression is an indeterminate form, we may apply L'Hospital's Rule to obtain

$$\text{Log } 10 \lim_{N\to\infty} \frac{\left\{(c_1+c_2)10^{[2(c_1+c_2)/\sqrt{N}]} - 2c_1 10^{[(2c_1)/\sqrt{N}]} + (c_1-c_2)10^{[2(c_1-c_2)/\sqrt{N}]}\right\}}{1/\sqrt{N}}$$

$$= \frac{0}{0}.$$

Applying L'Hospital's Rule for the second time

$$(\text{Log } 10)^2 \lim_{N\to\infty} \left\{ (c_1 + c_2)^2 \, 10^{[2(c_1+c_2)/\sqrt{N}]} - 2c_1^2 \, 10^{[(2c_1)/\sqrt{N}]} + (c_1 - c_2)^2 \, 10^{[2(2c_1-c_2)/\sqrt{N}]} \right\} = (\text{Log } 10)^2 \, c_2^2. \tag{5.8}$$

Thus if we examine (5.6), (5.7), and (5.8), we may now write

$$\lim_{N\to\infty} [NP(n \geq N)] = \frac{c(q_1,q_2)}{3} \left\{ \frac{3(\text{Log } 10)^2 \, c_2^2}{4[d(q_2,q_1)]^2} \right\}$$

$$= \frac{[(q_1+q_2-2)/2]!}{[(q_1-2)/2]! \, [(q_2-2)/2]!} \left(\frac{q_2}{q_1}\right)^{q_2/2} \frac{\sigma_1^2 (\text{Log } 10)^2 \, c_2^2}{4\sigma_2^2 \, dF_{1-\beta}(q_2,q_1)}, \tag{5.9}$$

where $c_2 = (y^2 + 3)/6$ and y is the normal deviate such that

$$\frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2} \, dx = \frac{\alpha}{2}.$$

Thus

$$\frac{-c(q_1,q_2)(\text{Log } 10)^2 \, c_2^2}{4[d(q_2,q_1)]^2} + 1 + \sum_{u=1}^{\infty} \int_{\frac{dF_{1-\beta}(m_2,m_1)\,\sigma_2^2}{C(u)F_{1-\beta}(u)\sigma_1^2}}^{\infty} W(x;m_2,m_1) \, dx$$

$$\leq E(k) \leq 1 + \sum_{u=1}^{\infty} \int_{\frac{dF_{1-\beta}(m_2,m_1)\,\sigma_2^2}{C(u)F_{1-\beta}(u)\sigma_1^2}}^{\infty} W(x;m_2,m_1) \, dx, \tag{5.10}$$

where $W(x;m_2,m_1)$ follows the F distribution with $m_2 - 1$ and $m_1 - 1$ degrees of freedom.

We shall show (5.10) converges by the comparison test. From (5.3) we may write

$$\frac{x^{(q_2-2)/2}}{\left[1 + (q_2/q_1)x\right]^{(q_1+q_2)/2}} \leq \frac{x}{(1 + x)^4}$$

for all x such that

$$x \geq c^{q_2(c+1)/(cq_2-4)} - c$$

if $q_1 > q_2$ or all $x > 0$ if $q_2 > q_1$.

Since

$$\frac{x}{(1 + x)^4} \leq \frac{1}{x^3},$$

for all $x \geq 0$, we may write

$$\sum_{u=u^*}^{\infty} \int_{\theta f_1(u)}^{\infty} W(x;q_2,q_1)\ dx \leq c(q_1,q_2) \sum_{u=u^*}^{\infty} \int_{\theta f_1(u)}^{\infty} \left(\frac{1}{x^3}\right) dx,$$

where u^* is the smallest integer such that

$$\frac{dF_\beta(q_2,q_1)\sigma_2^2}{C(u)F_{1-\beta}(u)\sigma_1^2} \geq \left\{ c^{[q_2(c+1)]/(cq_2-4)} - c \right\} .$$

We now examine

$$c(q_1,q_2) \sum_{u=u^*}^{\infty} \left[\frac{C(u)F_\beta(u)}{dF_{1-\beta}(q_2,q_1)}\right]^2 \tag{5.11}$$

for convergence. Use of the root test gives

$$\lim_{u \to \infty} F_\beta(u)[C(u)] = 0.$$

Since (5.11) converges, we may write

$$E(k) \leq 1 + \sum_{u=1}^{\infty} \int_{\theta f_1(u)}^{\infty} W(x;m_2,m_1)\, dx,$$

where

$$\theta f_1(u) = \frac{dF_\beta(q_2,q_1)\sigma_2^2}{F_{1-\beta}(u)C(u)\sigma_1^2}$$

and $W(x;m_2,m_1)$ is the F distribution with $m_2 - 1$ and $m_1 - 1$ degrees of freedom.

CHAPTER VI

ON THE ESTIMATE OF THE MEAN WITH DESIRED PRECISION

The expected value of n for a $1 - \alpha$ confidence interval and β^2 width coefficient on the mean of a normal distribution is given as

$$E(n) = 1 + \sum_{u=1}^{\infty} \int_{a(u)}^{\infty} W(x;m)\, dx,$$

where

$$W(x;m) = \frac{1}{2^{(m_1)/2}[(m_1 - 2)/2]!}\, x^{(m_1-2)/2} e^{-(x/2)};$$

$$m_1 = m - 1;$$

and

$$a(u) = \frac{u(u - 1)}{t^2_{\alpha/2}(u)\chi^2_{1-\beta}} \frac{\chi^2_{\beta}(m)}{4} \frac{d^2}{\sigma^2}.$$

The following computations were made through the use of the IBM 704 high speed digital computer at the Los Alamos Scientific Laboratory, Los Alamos, New Mexico. Each integral $>10^{-8}$ was evaluated and summed before multiplication by the constant $(1/2^{m_1/2})\ [1/(m_1 - 2)/2)]!$ which provided answers to a reasonable degree of accuracy. For example, if m_1 were to equal 20, the sum would include terms of order 10^{-17}.

Table III may be used to compare the expected value of n when using Stein's (4) two-stage method. Our results are compared with

TABLE III

EXPECTED VALUE OF n FOR A SPECIFIED WIDTH INTERVAL ON THE MEAN OF A NORMAL POPULATION ($m_1 = m - 1$)

m_1	d/σ											
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
					$\alpha = 0.05,\ \beta = 0.90$							
60	2061	536	248	146	98	71	55	44	36	31		
50	2118	551	256	150	100	73	56	45	37	31		
40	2199	572	265	156	104	76	58	46	38	32		
30	2324	604	280	164	110	79	61	49	40	34		
20	2562	665	307	180	120	87	66	53	43	37		
10	3258	842	387	225	149	107	82	65	53	45		
5	4894	1256	573	331	218	155	117	93	75	63	32	21
					$\alpha = 0.05,\ \beta = 0.95$							
60	2236	587	273	162	108	79	61	49	41	35		
50	2313	606	282	167	112	81	63	51	42	36		
40	2429	636	296	174	118	86	66	53	44	37		
30	2608	682	318	187	125	91	70	56	46	39		
20	2944	768	357	209	140	101	77	62	51	43		
10	4044	1048	483	282	187	135	102	81	67	56		
5	6863	1763	805	465	306	218	165	130	105	88	44	28
					$\alpha = 0.05,\ \beta = 0.99$							
60	2616	695	327	195	130	88	66	53	43	37		
50	2748	729	343	204	136	94	69	55	45	38		
40	2943	779	366	217	146	102	74	58	48	40		
30	3264	862	405	240	161	114	83	64	52	44		
20	3920	1031	481	284	190	136	100	77	62	51		
10	6252	1626	751	439	291	209	158	123	98	80		
5	14335	3678	1677	967	634	451	340	266	215	178	89	56
					$\alpha = 0.01,\ \beta = 0.90$							
60	3523	909	417	243	160	116	88	70	57	48		
50	3621	934	429	249	165	119	90	72	59	49		
40	3759	969	444	259	171	123	93	74	62	52		
30	3976	1025	470	273	181	130	99	79	64	54		
20	4384	1129	517	300	198	142	108	86	70	59		
10	5580	1431	653	377	248	177	134	106	86	72		
5	8396	2142	972	558	365	259	195	153	123	102	51	33
					$\alpha = 0.01,\ \beta = 0.95$							
60	3815	990	457	266	177	128	98	78	64	54		
50	3947	1024	472	276	183	132	101	80	66	55		
40	4146	1075	495	290	192	138	105	84	69	58		
30	4452	1153	531	310	205	148	113	90	74	62		
20	5029	1301	598	348	231	165	126	100	82	68		
10	6921	1780	814	471	310	222	168	132	108	90		
5	11770	3005	1364	784	513	364	273	214	173	143	71	45

TABLE III (continued)

m_1	d/σ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
					$\alpha = 0.01,\ \beta = 0.99$							
60	4445	1164	542	319	212	154	118	94	78	66		
50	4672	1222	568	333	223	161	123	99	82	69		
40	5007	1309	607	357	238	173	132	105	86	73		
30	5558	1450	672	394	262	189	144	115	95	80		
20	6684	1737	803	469	312	225	171	136	111	93		
10	10689	2756	1263	732	483	345	261	206	167	139		
5	24590	6273	2849	1631	1063	754	564	440	355	293	143	88
					$\alpha = 0.10,\ \beta = 0.90$							
60	1463	384	179	106	72	53	41	33	27	23		
50	1503	395	184	109	74	54	42	33	28	24		
40	1560	409	191	113	76	56	43	34	29	24		
30	1649	432	201	119	80	58	45	36	30	25		
20	1817	475	221	130	87	63	49	39	32	27		
10	2308	600	277	162	108	78	60	48	39	33		
5	3464	893	409	238	157	113	85	68	55	46	24	16
					$\alpha = 0.10,\ \beta = 0.95$							
60	1589	421	198	117	80	59	46	37	31	26		
50	1643	434	204	121	83	61	47	38	32	27		
40	1726	456	214	127	86	63	49	39	33	28		
30	1853	489	229	136	91	67	52	42	35	29		
20	2090	550	257	152	102	74	57	46	38	32		
10	2868	748	347	203	136	98	75	60	49	42		
5	4859	1254	576	334	221	158	120	95	77	64	33	22
					$\alpha = 0.10,\ \beta = 0.99$							
60	1862	499	237	143	97	71	57	46	38	33		
50	1956	524	249	149	101	76	59	48	40	34		
40	2095	560	265	158	108	80	62	50	42	36		
30	2323	619	293	175	119	87	68	55	45	39		
20	2788	739	348	206	139	102	79	63	53	45		
10	4438	1162	540	317	212	153	117	94	77	65		
5	10149	2615	1198	694	457	326	246	194	157	131	66	42

Seelbinder's evaluation (9) of the expected value of n from Stein's method. Let us choose $\alpha = 0.05$. Let G_β identify Graybill's method at the β probability level and let S denote Stein's technique.

In Table IV we have added the m samples in the first stage of Graybill's method to justify the comparison. The comparison shows that Stein's method would be far superior if an experimenter were in doubt as to which technique to use. However, it is important that if the variance changes, then Stein's method does not provide an exact confidence interval; hence, the comparison does not vitiate the usage of Graybill's method.

Let us illustrate how Table III might provide an answer as to the optimum choice of m that will minimize m + n. Suppose the experimenter wanted a 95 per cent confidence interval with probability 90 per cent that the width $<d$ units. Suppose further that he had evidence that d/σ was between 0.8 and 1.0. The following list of values of m + n will be an aid as to the choice of the first sample size.

m_1	d/σ			
	0.8	0.9	1.0	Av
5	99	81	69	83.00
10	76	64	56	65.33
20	74	64	58	65.33
30	80	71	65	72.00
40	87	79	73	79.67
50	96	88	82	88.67

The experimenter could choose m between 10 and 20 and feel that m + n would be minimized.

TABLE IV

COMPARISON BETWEEN GRAYBILL'S AND STEIN'S METHODS FOR EXPECTED SAMPLE SIZE FOR DESIRED WIDTH CONFIDENCE INTERVAL ON THE MEAN OF A NORMAL POPULATION

	d/σ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$m_1 = 60$										
S	400	100	62	61						
$G_{0.90}$	2122	597	309	207						
$G_{0.95}$	2297	648	334	223						
$G_{0.99}$	2677	756	388	256						
$m_1 = 50$										
S	403	101	53	52						
$G_{0.90}$	2169	602	307	201						
$G_{0.95}$	2364	657	333	218						
$G_{0.99}$	2799	780	394	255						
$m_1 = 40$										
S	408	102	48	42	41					
$G_{0.90}$	2240	613	306	197	145					
$G_{0.95}$	2470	677	337	215	159					
$G_{0.99}$	2984	820	407	258	187					
$m_1 = 30$										
S	417	104	47	32	31					
$G_{0.90}$	2355	635	311	195	141					
$G_{0.95}$	2639	713	349	218	156					
$G_{0.99}$	3295	893	436	271	192					
$m_1 = 20$										
S	435	109	49	29	22	21				
$G_{0.90}$	2583	686	328	201	141	108				
$G_{0.95}$	2965	789	378	230	161	122				
$G_{0.99}$	3941	1052	502	305	211	157				
$m_1 = 10$										
S	496	124	56	32	21	16	13	11		
$G_{0.90}$	3269	853	398	236	160	118	93	76		
$G_{0.95}$	4055	1059	494	293	198	146	113	92		
$G_{0.99}$	6263	1637	762	450	302	220	169	134		
$m_1 = 5$										
S	661	165	74	42	47	19	14	11	10	8
$G_{0.90}$	4900	1262	579	337	224	161	123	99	81	69
$G_{0.95}$	6869	1769	811	471	312	224	161	136	111	94
$G_{0.99}$	14341	3684	1683	973	640	457	346	262	221	184

CHAPTER VII

ON THE ESTIMATE OF THE VARIANCE WITH DESIRED PRECISION

The problem of obtaining an estimate of the variance of a normal population with a guaranteed precision has been solved. We shall discuss three different solutions — alike only in the respect of being two-step procedures.

1. BIRNBAUM AND HEALY'S METHOD. The technique described by Birnbaum and Healy (3) is essentially used for estimating σ^2 such that the variance of the estimator is less than or equal to a specified constant d.

If s^2, an unbiased estimate of σ^2, is computed from the first sample of size m, then

$$n = \frac{2s^4(m - 1)^2}{B(m - 3)(m - 5)} + 1.$$

If we want an interval estimate, Birnbaum and Healy's method is readily applicable. This technique provides an n such that the width of the interval is always $\leq d$ specified units. The main disadvantage of this method is the use of Tchebycheff's inequality, a rather conservative interval.

By Tchebycheff's inequality we have

$$P(\hat{\sigma}^2 - \frac{d}{2} < \sigma^2 < \hat{\sigma}^2 + \frac{d}{2}) \geq 1 - \frac{4B}{d^2}.$$

Let

$$\alpha = \frac{4B}{d^2},$$

then

$$B = \frac{\alpha d^2}{4},$$

and

$$n = \frac{8s^4(m - 1)^2}{\alpha d^2(m - 3)(m - 5)} .$$

Also, the expected value of n is given as

$$E(n) = 1 + \frac{2(m + 1)(m - 1)\sigma^4}{(m - 3)(m - 5)B}$$

$$= 1 + \frac{8(m + 1)(m - 1)}{(m - 3)(m - 5)\alpha} \left(\frac{\sigma^2}{d}\right)^2 .$$

Thus if $\alpha = 0.01$, $m = 20$, and $d/\sigma^2 = 1.0$, then

$$E(n) = 1 + \frac{8(21)(19)}{(17)(15)(0.01)}$$

$$= 1252.76.$$

2. GRAYBILL'S METHOD. If we desire a $1 - \alpha$ confidence interval on σ^2, and $\beta^2 \leq P(w \leq d) \leq 2\beta - \beta^2$, then from (5) we may select the smallest integer k so that

$$\chi^2_{1-\beta}(k) \left[\frac{1}{\chi^2_{1-\alpha/2}(k)} - \frac{1}{\chi^2_{\alpha/2}(k)}\right] \leq \frac{d\ \chi^2_\beta(m)}{(m - 1)s^2}.$$

We also have from our earlier work

$$E(n) = 1 + \sum_{u=1}^{\infty} \int_{f_1(u)/\sigma^2}^{\infty} w(\chi^2;m)\ d\chi^2,$$

where

$$f_1(u) = \frac{d\chi^2_\beta(m)}{\chi^2_{1-\beta}(u)\left[\frac{1}{\chi^2_{1-\alpha/2}(u)} - \frac{1}{\chi^2_{\alpha/2}(u)}\right]}$$

and $W(\chi^2;m)$ is the central Chi-square distribution with m - 1 degrees of freedom. Table V shows a comparison of expected values of n, the second sample, in Methods 1 and 2 for several values of α, β, d/σ^2, and m, where G_β indicates Method 2 at the β per cent level and B indicates Method 1.

From Table V when $\alpha = 0.01$, $m = 20$, $d/\sigma^2 = 1.0$ and $\beta = 0.99$, we obtain $E(n) = 152$. It can be seen from this comparison between Methods 1 and 2 that Method 2 is generally better. However, it would be more beneficial to use Method 1 if d/σ^2 were quite large and $1 - \alpha$ rather small.

3. EXPECTED WIDTH TECHNIQUE. Suppose we desire the expected width to be $\leq d$ with an exact confidence coefficient. The derivation is as follows:

A $1 - \alpha$ confidence interval is given by

$$P\left[\frac{s_2^2(n-1)}{\chi^2_{\alpha/2}(n)} < \sigma^2 < \frac{s_2^2(n-1)}{\chi^2_{1-\alpha/2}(n)}\right] = 1 - \alpha.$$

Thus

$$w = s_2^2(n-1)\,C(n),$$

where

$$C(n) = \frac{1}{\chi^2_{1-\alpha/2}(n)} - \frac{1}{\chi^2_{\alpha/2}(n)},$$

and

$$E(w) = \sigma^2\, E(n-1)\, C(n).$$

Let us choose n such that

$$(n-1)\,C(n) = \frac{d(m-3)}{s^2(m-1)} .$$

TABLE V

COMPARISON OF METHODS 1 AND 2

	d/σ^2								
	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
				$1 - \alpha = 0.99$					
m = 20									
B	4890	3397	2496	1911	1510	1224	545	307	136
$G_{0.90}$	206	152	118	96	80	68	39	27	18
$G_{0.95}$	272	201	156	126	105	90	50	35	23
$G_{0.99}$	469	344	267	216	179	152	84	57	35
m = 30									
B	4221	2932	2154	1650	1304	1056	470	265	119
$G_{0.90}$	170	126	99	80	67	58	34	24	16
$G_{0.95}$	215	160	125	102	85	73	42	30	20
$G_{0.99}$	334	248	194	158	132	113	64	45	29
m = 60									
B	3667	2547	1871	1433	1133	918	409	231	103
$G_{0.90}$	136	102	80	65	55	48	29	21	15
$G_{0.95}$	163	122	96	79	67	58	35	25	17
$G_{0.99}$	224	168	132	108	91	79	47	33	22
				$1 - \alpha = 0.95$					
m = 20									
B	979	681	500	383	303	246	110	63	29
$G_{0.90}$	126	93	73	60	50	43	25	18	12
$G_{0.95}$	168	125	98	80	67	57	33	24	15
$G_{0.99}$	291	216	169	137	115	98	56	39	24
m = 60									
B	735	511	375	288	228	185	83	47	22
$G_{0.90}$	85	64	51	42	36	31	19	14	10
$G_{0.95}$	102	77	62	51	44	38	23	17	12
$G_{0.99}$	143	108	86	71	61	54	33	24	16
				$1 - \alpha = 0.90$					
m = 20									
B	502	349	257	197	157	127	57	33	15
$G_{0.90}$	93	69	55	45	38	33	26	14	10
$G_{0.95}$	124	93	73	60	51	44	33	19	12
$G_{0.99}$	216	161	127	104	87	75	43	30	20
m = 60									
B	114	79	59	45	36	30	14	9	5
$G_{0.90}$	63	48	38	32	27	24	15	11	8
$G_{0.95}$	77	59	47	39	34	29	18	14	10
$G_{0.99}$	108	83	66	55	47	42	26	19	13

Then

$$E(w) = \sigma^2 \, E\left[\frac{d(m - 3)}{s^2(m - 1)}\right] = d.$$

Thus we need to find a value of k so that

$$(k - 1)\, C(k) \leq \frac{d(m - 3)}{s^2(m - 1)},$$

where k is the smallest integer $\geq n$.

A comparison between Methods 2 and 3 reveals that Method 3 is uniformly better. Let us examine the following relationships:

Method 2: $$\frac{\chi^2_{1-\beta}(n)\, C(n)}{\chi^2_{\beta}(m)} = x,$$

Method 3: $$\frac{(n - 1)\, C(n)}{(m - 3)} = x,$$

where $x = d/[s^2(m - 1)]$.

Since $(n - 1) < \chi^2_{1-\beta}(n)$ and $(m - 3) > \chi^2_{\beta}(m)$ for all n and m, then

$$\frac{(n - 1)}{(m - 3)} < \frac{\chi^2_{1-\beta}(n)}{\chi^2_{\beta}(m)}.$$

As x increases, n decreases; therefore, the inequality would imply that Method 3 is uniformly better.

We will now examine the expected value of n in Method 2. This is given as

$$E(n) = 1 + a \sum_{u=1}^{\infty} \int_{y(u)}^{\infty} W(x;m)\, dx,$$

where

$$W(x;m) = x^{[(m-2)/2]}\, e^{-(x/2)};$$

$$a = \frac{1}{2^{m/2}[(m - 2)/2]!};$$

and

$$y(u) = \frac{d\chi^2_\beta(m)}{\sigma^2\chi^2_{1-\beta}(u)\left[\frac{1}{\chi^2_{1-\alpha/2}(u)} - \frac{1}{\chi^2_{\alpha/2}(u)}\right]} .$$

Again, the problem of evaluation was set up for computation on an IBM 704 high speed digital computer. For convenience of evaluation, a transformation was made so that the constant in front of the summation could be written as $a_1 = 1/[(n - 2)/2]!$ Each integral greater than 10^{-8} was summed. This in itself may indicate the results are biased below their true values. However, the division by a_1 indicates that the evaluation included terms much smaller than 10^{-8}. For example when $m = 21$, the expected value included terms of the order 10^{-15}.

Table VI gives the expected value of n in Method 2. The following examples are given to illustrate possible uses of the table.

Example 1. Suppose an experimenter feels that d/σ^2 might lie between 0.8 and 1.5. Suppose further that he decides that $1 - \alpha$ equal to 0.95 is satisfactory, but he is unsure of β. The basic problem is to find the minimum value of $m + n$ such that the experimenter may select an optimum value of m. An additional problem is deciding what value of β to select. For instance, how many more samples would be necessary if β were equal to 0.99 instead of 0.95 or 0.90. The values of $m + n$ in Table VII were extracted from Table VI and might provide an answer to the problems.

We shall choose the minimum average value and select the corresponding value of m. Thus for $\beta = 0.90$, we choose $m_1 = 20$ and expect to take n between 25 and 60. For $\beta = 0.95$, we choose $m_1 = 30$ and expect

TABLE VI

EXPECTED VALUE OF n FOR SPECIFIED WIDTH OF CONFIDENCE INTERVAL ON THE VARIANCE OF A NORMAL POPULATION

m - 1	d/σ^2 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
						$1 - \alpha = 0.99,\ \beta = 0.99$							
5											839	503	252
10	24467	6517	3030	1767	1172	843	641	507	414	346	179	116	65
20	9174	2446	1157	691	469	344	267	216	179	152	84	57	35
30	6207	1680	805	487	334	248	194	158	132	113	64	45	29
40	4995	1363	658	401	278	208	164	134	112	96	56	40	26
50	4332	1190	578	353	245	183	145	119	101	87	51	36	24
60	3916	1079	526	323	224	168	132	108	91	79	47	33	22
						$1 - \alpha = 0.99,\ \beta = 0.95$							
5												134	72
10						368	282	225	185	155	83	55	33
20				400	272	201	156	126	105	90	50	35	23
30			518	313	215	160	125	102	85	73	42	30	20
40			450	274	189	141	111	90	76	67	38	28	19
50		846	408	250	173	130	102	84	71	62	36	26	18
60			383	234	163	122	96	79	67	58	35	25	17
						$1 - \alpha = 0.99,\ \beta = 0.90$							
5												73	41
10						243	187	149	123	104	56	38	23
20				303	206	152	118	96	80	68	39	27	18
30			411	248	170	126	99	80	67	58	34	24	16
40				224	154	115	90	73	61	53	32	23	15
50				208	144	107	84	68	58	50	30	22	15
60			321	197	136	102	80	65	55	48	29	21	15
						$1 - \alpha = 0.95,\ \beta = 0.99$							
5									1286	1059	510	309	158
10						514	393	313	257	216	114	75	44
20				424	291	216	169	137	115	98	56	39	24
30			493	302	210	157	124	102	86	74	43	31	20
40			405	251	175	132	105	86	73	63	38	27	18
50			361	225	157	118	93	77	66	58	35	25	17
60			327	204	143	108	86	71	61	54	33	24	16
						$1 - \alpha = 0.95,\ \beta = 0.95$							
5									319	265	133	84	46
10	6269	1655	775	459	309	225	174	139	115	97	52	36	22
20	3115	843	404	245	168	125	98	80	67	57	33	24	15
30	2373	650	316	193	134	100	79	65	55	47	28	20	14
40	2038	563	275	169	118	89	70	58	49	43	26	19	13
50	1844	511	251	156	109	82	65	54	46	40	24	18	12
60	1716	478	235	145	102	77	62	51	44	38	23	17	12

TABLE VI (continued)

m - 1	d/σ^2 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
						$1 - \alpha = 0.95, \beta = 0.90$							
5									169	141	72	47	27
10						149	115	92	76	65	36	25	16
20				184	126	93	73	60	50	43	25	18	12
30			249	152	105	78	62	51	43	37	22	16	11
40			223	137	94	71	56	46	39	34	21	15	11
50			208	128	89	67	53	44	37	32	20	14	10
60			196	122	85	64	51	42	36	31	19	14	10
						$1 - \alpha = 0.90, \beta = 0.99$							
5									930	768	374	229	119
10						377	290	232	191	162	87	58	34
20				313	216	161	127	104	87	75	43	30	20
30				225	157	119	94	78	66	57	34	24	16
40			299	187	132	100	80	66	57	49	30	22	15
50			265	166	117	89	72	60	51	45	27	20	14
60			242	153	108	83	66	55	47	42	26	19	13
						$1 - \alpha = 0.90, \beta = 0.95$							
5									233	194	99	63	35
10						166	128	103	86	73	40	28	17
20				180	124	93	73	60	51	44	26	19	12
30			231	143	99	75	60	49	42	36	22	16	11
40			202	125	88	67	53	44	38	33	20	15	10
50			184	115	81	62	50	41	35	31	19	14	10
60			173	108	77	59	47	39	34	29	18	14	10
						$1 - \alpha = 0.90, \beta = 0.90$							
5									124	104	54	35	21
10	2916	776	367	219	149	109	85	69	57	49	28	19	12
20	1679	458	221	135	93	69	55	45	38	33	26	14	10
30	1352	373	182	112	78	59	46	38	33	28	17	13	9
40	1198	332	163	101	70	53	43	35	30	26	16	12	8
50	1107	308	152	94	66	50	40	33	28	25	16	12	8
60	1046	292	144	90	63	48	38	32	27	24	15	11	8

TABLE VII

EXPECTED VALUES OF $m_1 + n$ IF d/σ^2 LIES BETWEEN 0.8 AND 1.5, $\alpha = 0.05$, AND β LIES BETWEEN 0.90 AND 0.99 (FOR EXAMPLE 1)

m_1	d/σ^2 0.8	0.9	1.0	1.5	Av
			$1 - \alpha = 0.95$, $\beta = 0.90$		
10	102	86	75	46	77.25
20	80	70	63	45	64.50
30	81	73	67	52	68.25
40	86	79	74	61	75.00
50	94	87	82	72	83.75
60	102	96	91	79	92.00
			$1 - \alpha = 0.95$, $\beta = 0.95$		
10	149	125	107	62	110.75
20	100	87	77	53	79.25
30	95	85	77	58	78.75
40	98	89	83	66	84.00
50	104	96	90	74	91.00
60	111	104	98	83	91.50
			$1 - \alpha = 0.95$, $\beta = 0.99$		
10	323	267	226	124	235.00
20	157	135	118	76	121.50
30	132	116	104	73	106.25
40	126	113	103	78	105.00
50	127	116	108	85	109.00
60	131	121	104	93	112.25

to take n between 28 and 65. For $\beta = 0.99$, we choose $m_1 = 40$ and expect to take n between 38 and 86.

Example 2. If d/σ^2 was thought to lie between 0.8 and 1.0, $\beta = 0.90$, $1 - \alpha = 0.90$, then the choice of the initial sample might be found from the following table of m + n values.

m-1	d/σ^2			
	0.8	0.9	1.0	av
5	--	130	110	120
10	80	68	60	70
20	66	59	54	60
30	69	64	59	64

Thus an initial sample of size 20 would minimize the total sample size.

CHAPTER VIII

ON THE RECTANGULAR DENSITY

We shall devote this chapter to the rectangular distribution obtaining (1) a two-step procedure for a specified width interval estimate of the parameter, (2) a one-step procedure for a specified width interval estimate, and (3) the expected sample size in the two-step procedure.

A two-step procedure for a specified width interval estimate on the parameter of a rectangular density is given as follows:

Let

$$f(x) = \frac{1}{\theta}; \qquad 0 < x < \theta.$$

We desire $1 - \alpha$ confidence interval on θ, and $P(w < d) \geq \beta^2$. Let us denote

$$y = \max(x_1, x_2 \cdots x_n).$$

Then the width of the interval may be written

$$w = y\left(\frac{1}{\alpha^{1/n}} - 1\right).$$

Let

$$Y = \frac{w}{\theta\left(\frac{1}{\alpha^{1/n}} - 1\right)} = g(w; \theta, n).$$

The probability density function of Y is given by

$$F(Y) = nY^{n-1}; \qquad 0 < Y < 1.$$

The density of Y is independent of any unknown parameters. Let

$$f(n) = \beta^{1/n}.$$

Then

$$P[Y < f(n)] = \int_0^{\beta^{1/n}} nY^{n-1}\, dY = \beta.$$

We may also write

$$P\left\{ \frac{w}{\theta[(1/\alpha^{1/n})-1]} < f(n) \right\} = \beta.$$

Let $g(w;\theta,n) = f(n)$ and solve for w. We obtain

$$w = h(\theta;n) = \beta^{1/n}\, \theta\left(\frac{1}{\alpha^{1/n}} - 1\right). \tag{8.1}$$

We observe that

$h(\theta;n)$ is monotonically increasing in θ, (8.2a)

and

$h(\theta;n)$ is monotonically decreasing in n. (8.2b)

Suppose in step one of the procedure we take a sample of size m and let z equal the maximum value. Let

$$t(z) = qz, \tag{8.3}$$

where

$$q = \left(\frac{1}{1 - \beta}\right)^{1/m}.$$

Then

$$P[t(z) > \theta] = P(z > \frac{\theta}{q}) = \int_{\theta/q}^{\theta} \left[\frac{mz^{m-1}}{\theta^m}\right] dz = \beta. \tag{8.4}$$

When $\theta = t(z)$ we have

$$h[t(z),n] = \beta^{1/n} qz \left(\frac{1}{\alpha^{1/n}} - 1\right).$$

By (8.1) and (8.2a) we may write

$$P[w < h(\theta_1,n) \mid \theta_1 > \theta] \geq P[w < h(\theta,n)]$$

$$= P\left[y\left(\frac{1}{\alpha^{1/n}} - 1\right) < \beta^{1/n}\theta\left(\frac{1}{\alpha^{1/n}} - 1\right)\right]$$

$$= P\left(\frac{y}{\theta} < \beta^{1/n}\right)$$

$$= P(Y < \beta^{1/n}) = \beta. \tag{8.5}$$

By (8.4) and (8.5) we obtain

$$P(w < d) \geq P[w < d, t(z) > \theta] = P\{w < h[t(z),n] \mid t(z) > \theta\} P[t(z) > \theta] \geq \beta$$

Since we set $h[t(z),n] = d$, we choose the smallest integral value of $n = k$ so that

$$\beta^{1/k}\, qz\left(\frac{1}{\alpha^{1/k}} - 1\right) \leq d.$$

Table VIII is provided for

$$\frac{d}{qz} = Q_k = \beta^{1/k}\left(\frac{1 - \alpha^{1/k}}{\alpha^{1/k}}\right).$$

For example, suppose a 99 per cent confidence interval is desired on θ with $P(w \leq 1) \geq 0.81$. We have $\beta = 0.90$, $1 - \alpha = 0.99$, and $d = 1$. Suppose we compute $qz = 5$. Then $Q_n = 0.20$ and a sample of size 26 will meet the requirements.

We shall now describe a one-step procedure whereby a value of n

TABLE VIII

VALUES NECESSARY FOR DETERMINING n WHEN SETTING A SPECIFIED WIDTH INTERVAL ON THE PARAMETER OF THE RECTANGULAR DISTRIBUTION

n	Qn		
	$\beta = 0.90$ $\beta^2 = 0.81$ $2\beta - \beta^2 = 0.99$	$\beta = 0.95$ $\beta^2 = 0.9025$ $2\beta - \beta^2 = 0.9975$	$\beta = 0.99$ $\beta^2 = 0.9801$ $2\beta - \beta^2 = 0.9999$
		$\boxed{1 - \alpha = 0.99}$	
1	89.100	94.050	98.010
2	8.538	8.772	8.955
3	3.515	3.580	3.629
4	2.106	2.135	2.157
5	1.480	1.496	1.509
6	1.134	1.145	1.153
7	0.917	0.924	0.929
8	0.768	0.773	0.777
9	0.660	0.664	0.667
10	0.579	0.582	0.584
20	0.257	0.258	0.258
30	0.165	0.166	0.166
40	0.122	0.122	0.122
50	0.096	0.096	0.096
60	0.080	0.080	0.080
100	0.047	0.047	0.047
200	0.023	0.023	0.023
500	0.009	0.009	0.009

TABLE VIII (continued)

n	Qn		
	$\beta = 0.90$ $\beta^2 = 0.81$ $2\beta - \beta^2 = 0.99$	$\beta = 0.95$ $\beta^2 = 0.9025$ $2\beta - \beta^2 = 0.9975$	$\beta = 0.99$ $\beta^2 = 0.9801$ $2\beta - \beta^2 = 0.9999$
		$\boxed{1 - \alpha = 0.95}$	
1	17.100	18.050	18.810
2	3.293	3.384	3.454
3	1.655	1.685	1.709
4	1.086	1.100	1.112
5	0.803	0.812	0.819
6	0.636	0.642	0.646
7	0.526	0.530	0.533
8	0.448	0.451	0.454
9	0.390	0.393	0.395
10	0.346	0.347	0.349
20	0.161	0.161	0.162
30	0.105	0.105	0.105
40	0.078	0.078	0.078
50	0.062	0.062	0.062
60	0.051	0.051	0.051
100	0.030	0.030	0.030
200	0.015	0.015	0.015
500	0.006	0.006	0.006

TABLE VIII (continued)

n	Qn		
	$\beta = 0.90$ $\beta^2 = 0.81$ $2\beta - \beta^2 = 0.99$	$\beta = 0.95$ $\beta^2 = 0.9025$ $2\beta - \beta^2 = 0.9975$	$\beta = 0.99$ $\beta^2 = 0.9801$ $2\beta - \beta^2 = 0.9999$
		$\boxed{1 - \alpha = 0.90}$	
1	8.100	8.550	8.910
2	2.051	2.108	2.151
3	1.114	1.135	1.151
4	0.758	0.768	0.776
5	0.573	0.579	0.584
6	0.460	0.464	0.467
7	0.384	0.387	0.389
8	0.329	0.331	0.333
9	0.288	0.290	0.291
10	0.256	0.258	0.259
20	0.121	0.122	0.122
30	0.079	0.080	0.080
40	0.059	0.059	0.059
50	0.047	0.047	0.047
60	0.039	0.039	0.039
100	0.023	0.023	0.023
200	0.012	0.012	0.012
500	0.005	0.005	0.005

may be found for a specified $d = p\theta/100$ with confidence coefficient $1 - \alpha$ and width coefficient β where $0 < \beta < 1$. The technique is derived in the following manner.

$$P(w \leq \frac{p}{100}\,\theta) = \beta\,,$$

$$P\left\{\frac{y}{\theta} \leq \frac{p}{100}\,\frac{1}{[(1/\alpha^{1/n}) - 1]}\right\} = \beta\,,$$

$$\int_{o}^{p/100} \left[\frac{1}{(\alpha^{1/n} - 1)}\right] nx^{n-1}\,dx = \beta.$$

Hence

$$\frac{p}{100} = \frac{\beta^{1/n}}{\alpha^{1/n}}\,(1 - \alpha^{1/n}).$$

Thus we find the smallest integral value of $n = k$ so that

$$\frac{p}{100} \geq \frac{\beta^{1/k}(1 - \alpha^{1/k})}{\alpha^{1/k}}.$$

For example, suppose a 95 per cent confidence interval on θ is desired so that the width <10 per cent of θ, with width coefficient of 0.99. Therefore: $p/100 = 0.10$, $\beta = 0.99$, and $1 - \alpha = 0.95$. Table VIII shows that a sample of size 31 will suffice.

The expected value of k in this technique may be derived quite readily. From an earlier proof

$$E(k) = \lim_{N \to \infty} \sum_{u=1}^{N-1} P(n \geq u) + 1 - \lim_{N \to \infty} NP(n \geq N).$$

Let

$$z = f_1(n) = \frac{d}{q\beta^{1/n}[(1/\alpha^{1/n}) - 1]}.$$

The value of z is maximized when $z = \theta$. Let N be a value of n such that

$$\int_{f_1(N)=\theta}^{\infty} g(z)\ dz = 0, \tag{8.6}$$

where $g(z)$ is the density function of z. Since $f_1(n)$ is monotonically increasing in n, then

$$P(n \geq N) = P[f_1(n) \geq f_1(N)].$$

From (8.6)

$$\lim_{N \to \infty} NP(n \geq N) = \lim_{N \to \infty} N \int_{f_1(N)}^{\infty} g(z)\ dz = 0.$$

Therefore

$$E(k) = 1 + \sum_{u=1}^{u^*} \int_{f_1(u)}^{\theta} g(z)\ dz,$$

where

$$g(z) = \frac{mz^{m-1}}{\theta^m} ;$$

u^* is the smallest integer such that $f_1(u^*) \geq \theta$;

$$f_1(u) = \frac{d}{q\ \beta^{1/u}[(1/\alpha^{1/u}) - 1]} ;$$

and

$$q = \left(\frac{1}{1 - \beta}\right)^{1/m} .$$

Evaluating the integral we obtain

$$E(k) = 1 + \sum_{u=1}^{u^*} \left\{ 1 - \left[\left(\frac{d}{\theta}\right)^m \left(\frac{1}{q\ \beta^{1/u}[(1/\alpha^{1/u}) - 1]}\right)^m \right] \right\}.$$

Solutions for various values of m for the expected value of k are given in Table IX.

TABLE IX

EXPECTED VALUE OF k FOR SPECIFIED WIDTH INTERVAL ON THE PARAMETER OF THE RECTANGULAR DISTRIBUTION

n	d/θ									
	0.05	0.06	0.07	0.08	0.09	0.10	0.20	0.30	0.40	0.50
					$1 - \alpha = 0.95,\ \beta = 0.99$					
10	89.31	74.92	64.64	56.93	50.93	46.12	24.53	17.34	13.74	11.50
20	74.80	62.84	54.30	47.88	42.90	38.88	20.94	14.78	11.83	9.92
30	70.59	59.32	51.28	45.20	40.54	36.80	19.76	14.12	11.38	9.81
40	68.57	57.66	49.85	44.00	39.39	35.71	19.41	13.90	10.98	9.51
50	67.38	56.66	49.02	43.27	38.66	35.21	18.93	13.82	10.98	9.00
60	66.61	56.05	48.45	42.67	38.39	34.75	18.90	13.69	10.98	9.00
					$1 - \alpha = 0.95,\ \beta = 0.95$					
10	76.43	64.17	55.42	48.85	43.76	39.67	21.30	15.12	12.11	10.27
20	69.20	58.18	50.27	44.33	39.73	36.07	19.46	13.81	11.09	9.60
30	67.02	56.34	48.70	42.97	38.58	35.00	18.81	13.68	10.91	8.99
40	65.89	55.44	47.96	42.30	37.96	34.44	18.73	13.42	10.90	8.99
50	65.26	54.95	47.44	41.91	37.57	34.13	18.61	12.99	10.87	8.99
60	64.92	54.50	47.23	41.63	37.41	33.80	18.43	12.99	10.85	8.99
					$1 - \alpha = 0.91,\ \beta = 0.99$					
10	136.48	114.36	98.57	86.72	77.49	70.12	36.92	25.85	20.31	16.94
20	114.21	95.79	82.64	72.79	65.12	58.98	31.35	22.15	17.51	14.69
30	107.70	90.38	77.97	68.73	61.47	55.69	29.67	20.98	16.72	13.92
40	104.61	87.81	75.76	66.76	59.79	54.13	28.87	20.60	16.22	13.84
50	102.79	86.30	74.52	65.68	58.80	53.27	28.56	20.05	15.96	13.71
60	101.60	85.28	73.66	64.94	58.15	52.58	28.12	19.94	15.95	13.42
					$1 - \alpha = 0.90,\ \beta = 0.90$					
10	55.23	46.47	40.19	35.48	31.82	28.86	15.73	11.32	9.12	7.74
20	51.76	43.53	37.66	33.31	29.88	27.11	14.73	10.76	8.76	7.46
30	50.66	42.67	36.94	32.59	29.29	26.54	14.65	10.66	8.64	7.00
40	50.12	42.21	36.49	32.33	28.95	26.39	14.49	10.50	8.47	7.00
50	49.83	41.95	36.36	32.09	28.76	26.16	14.26	10.25	8.20	7.00
60	49.56	41.69	36.18	31.83	28.73	25.90	13.99	10.00	8.00	7.00

In order to select an m to minimize the total sample size, the experimenter would only need to choose m = 10 or 20 to obtain the desired results. For example, if d/θ was thought to lie between 0.09 and 0.20, $\beta = 0.99$, $\alpha = 0.05$, then the following list illustrates that m = 10 would be a desirable choice.

m	d/θ		
	0.09	0.10	0.20
10	60.93	56.12	34.53
20	62.90	58.88	40.94
30	70.54	66.80	49.76

CHAPTER IX

SUMMARY

Summarizing, we have enlarged upon the technique developed by Graybill and made comparisons with work by Stein and Birnbaum and Healy. We have found the following:

1. Bounds on the confidence coefficient,

2. Tables for the expected value of the sample size necessary for a specified width interval on the mean of a normal population,

3. Tables for the expected value of the sample size necessary for a specified width interval on the variance of a normal population,

4. A method for finding the sample size necessary for a $1 - \alpha$ confidence interval and a β^2 width coefficient on the ratio of variances from two normal populations,

5. Expected value of the sample size necessary for a specified width interval on the ratio of variances from two normal populations,

6. A method for finding the sample size necessary for a $1 - \alpha$ confidence interval and β^2 width coefficient on the parameter of the rectangular density, and

7. Tables for the expected value of the sample size necessary for a specified width interval on the parameter of the rectangular distribution.

Further work in this area might include:

1. Finding tables for the expected value of the sample size necessary for a specified width interval on the ratio of variances,

2. Finding a method for the sample size necessary for a specified

width interval on the intra-class correlation coefficient $\sigma_1^2/(\sigma_1^2 + \sigma_2^2)$,

3. Investigating methods to minimize the first sample size, and

4. Investigating a method to determine the necessary sample size for specified width interval on a regression line.

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